Essential Mathematics for Engineering Technicians

The National Center for Optics and Photonics Education

OP-TEC
National Center for Optics and Photonics Education

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Preface

Mathematics skills are essential for satisfactory progress in learning the technical content of material found in typical two-year engineering technology education programs. Students entering such programs come with varying backgrounds of mathematics achievement and varying retention levels of mathematics once learned.

To help aspiring engineering technicians begin their studies with adequate math skills, OP-TEC has developed an ebook—*Mathematics for Engineering Technicians*. This text covers the following topics:

- Scientific Notation
- Unit Conversion
- Introductory Algebra
- Powers and Roots
- Ratio and Proportion
- Exponents and Logarithms
- Graphing in Rectangular Coordinates
- Geometry
- Angle Measures in Two and Three Dimensions
- Trigonometry
- Special graphs
- Sinusoidal Motion
- Complex Numbers

The material in Mathematics for Engineering Technicians is intended for use by ET faculty to assist their students in understanding technical concepts and acquire problem-solving skills. It is not intended to be a text for a math course. The text can be used in several ways:

- The material can be presented as a whole in a “regular class” with teacher lecture and practice sessions, generally not for credit.
- The material can be assigned for appropriate students to “learn on their own” with occasional instructor monitoring and help.
- The material can be used “piecemeal” in a self-learning mode—with some instructor help—by students who require review and brush up in one or more specific mathematics areas.
- The material can be used as a student and faculty reference for math-based ET classes

Student Assessment

An assessment instrument containing multiple-choice questions and titled *Entering Student Assessment for Mathematics for Engineering Technicians* is also available for faculty to help identify specific topics individual students need extra help with. To obtain the assessment contact:

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Features

Objectives & Scenario Question

The Objectives section contains the learning objectives for each chapter. It also contains the Scenario question for each chapter.

Basic Concepts

The Basic Concepts section contains the lesson content, videos, diagrams, tables, graphs, and interactive widgets for each chapter. Within each chapter, videos are provided to improve the understanding of presented concepts.

Solution to Scenario Question

This section contains the solution to the Scenario question presented in the beginning of each chapter.

Practice Exercises

This section contains exercises and questions related to the chapter's lesson.

Solutions to Practice Exercises

This section contains the solutions to the Practice Exercises.

Mathematics Videos

Math videos are provided to assist students in performing mathematical operations required in the text. The videos that are in the ebook are also available at: http://optecvideo.opteccrm.org.

Real World Connections and Applications:

Students are interested in how engineering technicians prepare for their careers, what skills they will need to be successful in future jobs, and what kinds of responsibilities they might be assigned in the workplace. The text has two features that address these questions and bring real world context to its content.

Scenario Questions

Each chapter begins with a Scenario question that provides a connection between the chapter material and how these technicians can apply it. Solutions to the Scenario questions are provided within the chapter.

Career Videos

This feature contains videos of various careers in engineering technologies such as manufacturing, electronics, lasers, optics, research, energy and more.

Scientific Notation

1

Objectives

1. Write decimal numbers in scientific notation
2. Write scientifically notated numbers in decimal notation

Scenario

A Type J thermocouple uses a calibration equation of the form:

$$T = a_0 + a_1V + a_2V^2 + a_3V^3 + L + a_nV^n$$

The coefficients a_0, a_1, etc. are listed as shown in the computer printout to the right as A0, A1, etc. for temperatures between 0 and 760°C (or voltages between 0 and 42.919 mV). V is the measured voltage in millivolts (mV). Note that computers show scientific notation values like 1.5×10^6 as "1.5E+06."

```
Type J: 0 to 760 degC
  A0: 0.000000E+00
  A1: 1.978425E+01
  A2: –2.001204E–01
  A3: 1.036969E–02
  A4: –2.549687E–04
  A5: 3.585153E–06
  A6: –5.344285E–08
  A7: 5.099890E–10
```

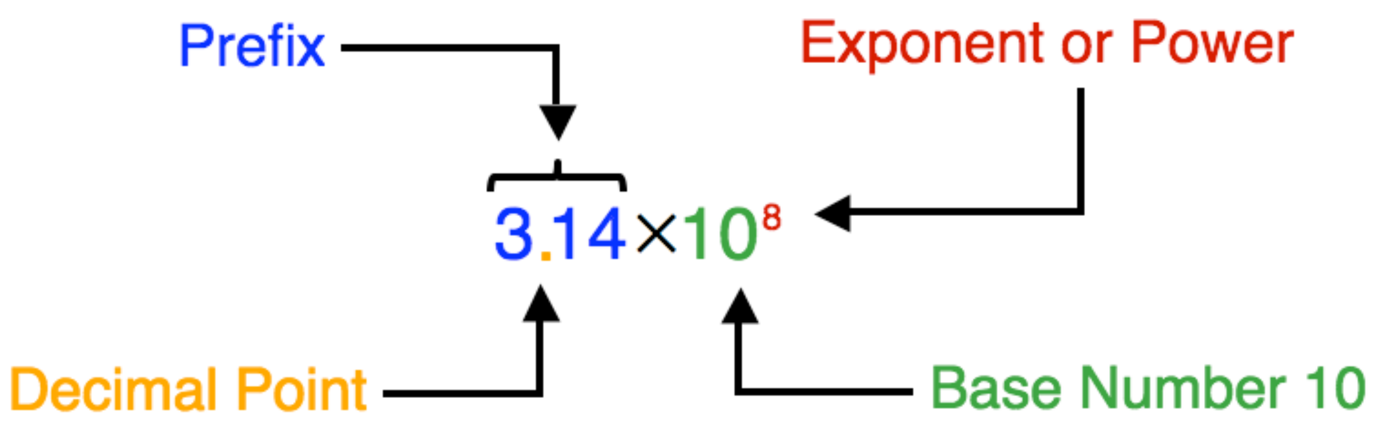
Write the calibration equation for a Type J thermocouple used from 0 to 760°C expressing the coefficients in scientific notation.

What temperature is indicated by a thermocouple measurement of 18.3 mV?

Before looking at the solution, work through the lesson to further develop your skills in this area.

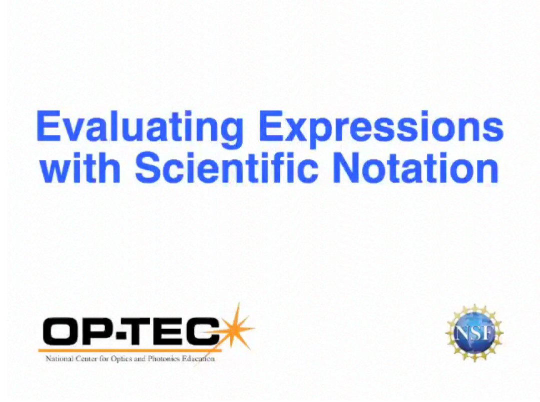
The Basics of Scientific Notation

Scientific notation makes use of "powers of ten." It is a convenient way to write large or small numbers. For example, the large number 314,000,000,000 becomes 3.14×10^{11} in scientific notation. The small number 0.000000314 becomes 3.14×10^{-7}. Before we learn how to write ordinary decimal numbers in scientific notation, let's learn a few definitions. Scientific notation involves a prefix, a decimal point, the number 10, and a number called an "**exponent**," or **power**. These are identified below.



Any one of the numbers in the prefix—3, 1, or 4—is referred to as a digit.

Movie 1.1 Evaluating Expressions with Scientific Notation



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Scientific notation is based on the fact that you can write any number as a prefix times a string of tens. For example, a string of four tens— $10 \times 10 \times 10 \times 10$ —can be written as 10^4. Study the following examples.

$$100 = 1 \times 100 = 1 \times (10 \times 10) = 1 \times 10^2$$
$$150 = 1.5 \times 100 = 1.5 \times (10 \times 10) = 1.5 \times 10^2$$
$$3750 = 3.75 \times 1000 = 3.75 \times (10 \times 10 \times 10) = 3.75 \times 10^3$$
$$37{,}500 = 3.75 \times (10 \times 10 \times 10 \times 10) = 3.75 \times 10^4$$

Note that, in each case, a number (such as 3750) was written as a prefix (3.75) times a power of ten (10^3).

When a number is less than one, such as 0.1 or 0.015, it can also be written in scientific notation. For example:

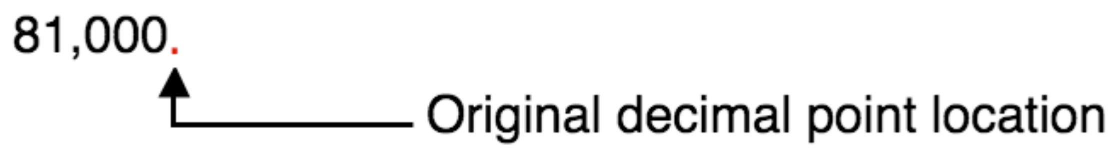
$$0.1 = 1 \times \frac{1}{10} = 1 \times 10^{-1}; \left(\frac{1}{10} = 10^{-1}\right)$$
$$0.01 = 1 \times \frac{1}{100} = 1 \times \frac{1}{10 \times 10} = 1 \times 10^{-2}; \left(\frac{1}{10 \times 10} = \frac{1}{10^2} = 10^{-2}\right)$$
$$0.015 = 1.5 \times \frac{1}{100} = 1.5 \times \frac{1}{10 \times 10} = 1.5 \times 10^{-2}$$

Note that, in each case, a number less than one was written as a prefix times a power of ten. For a number less than one, the exponent or power used with the number 10 is always **negative**.

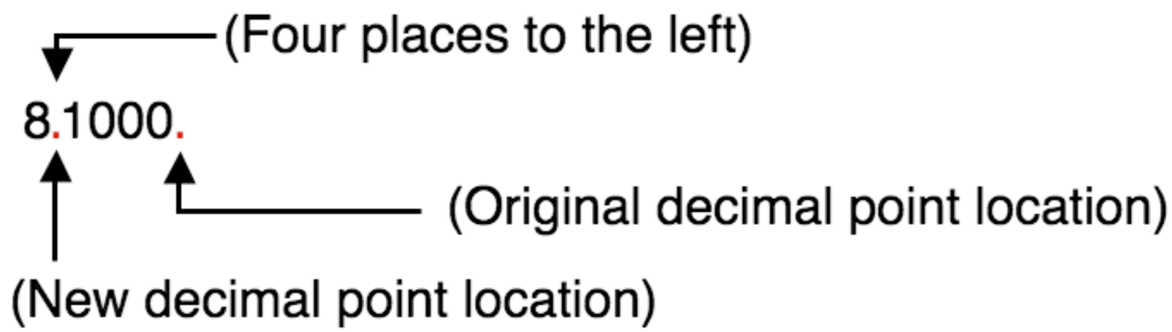
When a number such as 81,000 is written as 8.1×10^4, it's called "**scientific notation**." It's called that because scientists, engineers, and technicians often have to use very large or very small numbers. Writing one of these numbers as a prefix times a power of ten is easier and faster. To write numbers in scientific notation, follow these procedures.

Case 1—The general procedure for writing numbers greater than one (such as 81,000) in scientific notation is:

a. Locate the decimal point in the number.



b. Count the number of places to the left to shift the decimal point so that only **one digit** remains to the left of the new decimal point.

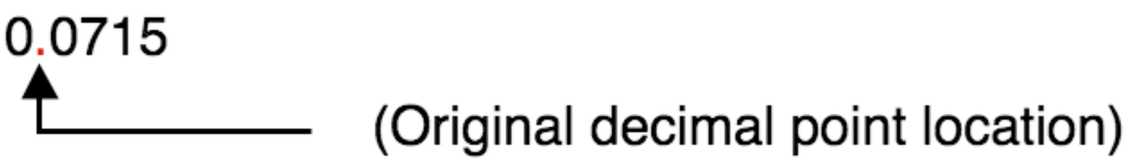


c. Use the number of places moved to the left (4) as the **positive exponent** in the power of ten (10) and write 81,000 in scientific notation as:

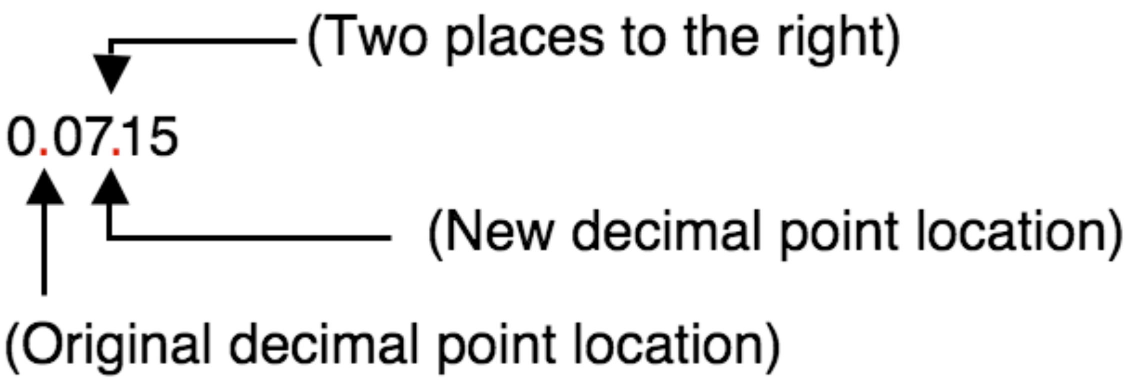
$$81{,}000 = 8.1 \times 10^4$$

Case 2—The general procedure for writing **numbers less than one** (such as 0.0715) in scientific notation is as follows:

a. Locate the decimal point in the number.



b. Count the number of places to the right to shift the decimal point so that only **one digit**, other than zero, remains to the left of the new decimal point.



c. Use the number of places moved to the right (2) as the negative exponent in the power of ten (10^{-2}) and write 0.0715 in scientific notation as:

$$0.0715 = 7.15 \times 10^{-2}$$

Table 1.1 Converting Numbers to Power-of-Ten Notation

375,000	=	3.75×10^{5}
81,000	=	8.1×10^{4}
623,000,000,000	=	6.23×10^{11}
0.0715	=	7.15×10^{-2}
0.0025	=	2.5×10^{-3}
0.000000133	=	1.33×10^{-7}

In Table 1.1, the procedures outlined above have been used to change ordinary decimal numbers to numbers in scientific notation. Look over each number and be sure that you agree with the conversion.

When a number such as 81,000 is changed to scientific notation, it's written as 8.1×10^{4} . Other forms of power-of-ten notation, equally correct and equally useful but not written in scientific notation, are the following:

$$\begin{aligned} 81{,}000 &= 81 \times 10^{3} \\ &= 810 \times 10^{2} \\ &= 8100 \times 10^{1} \\ &= 0.81 \times 10^{5} \end{aligned}$$

Similarly, for numbers less than one, more than one form for the power-of-ten notation is possible. For example:

$$\begin{aligned} 0.0025 &= 2.5 \times 10^{-3} \\ &= 25 \times 10^{-4} \\ &= 250 \times 10^{-5} \\ &= 0.25 \times 10^{-2} \end{aligned}$$

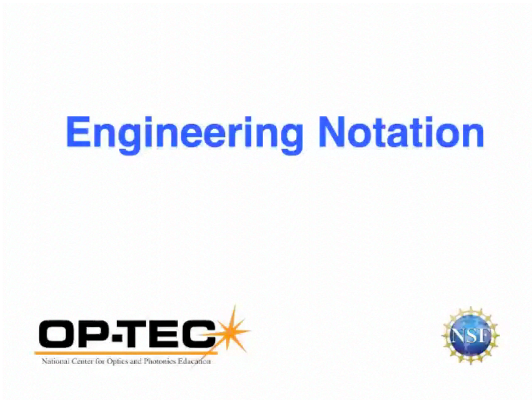
A specific form of power-of-ten notation that is sometimes used is called **engineering notation**, though it is not as common as scientific notation. It is important to understand the differences between the two. For engineering notation, the exponent must be a multiple of 3 (…, –12, –9, –6, –3, 3, 6, 9, 12, …) or zero, whereas for scientific notation it is necessary only that the exponent be an integer (…, –3, –2, –1, 0, 1, 2, 3, …).

TABLE 1.2 Equivalent Numbers in Scientific Notation and Engineering Notation

	Scientific Notation	Engineering Notation
a	7.2×10^{6}	7.2×10^{6}
b	3.5×10^{4}	35×10^{3}
c	8.4×10^{8}	840×10^{6}
d	9.1×10^{-5}	91×10^{-6}
e	6.0×10^{-7}	600×10^{-9}
f	4.2×10^{2}	420×10^{0}
g	5.7×10^{-21}	5.7×10^{-21}

In Table 1.2 are examples of numbers expressed both in scientific notation and engineering notation.

MOVIE 1.2 Engineering Notation



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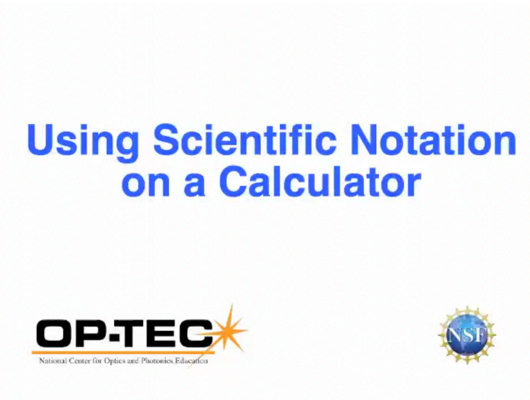
Do you see the pattern? When an exponent is not a multiple of 3 or a zero (as in letters b–f above), it is rounded down to the next multiple of 3 or zero. If moved, it will always be either one (b, d) or two places down (c, e, f). Then it is necessary to move the decimal place of the prefix (number before the “ × ”) to the right the same number of times as places moved down in the previous step. So, for letters b and d, the decimal in the prefix is moved one place to the right (essentially multiplying the prefix by 10) and, for letters c, e, and f, the decimal is moved two places to the right (like multiplying by 100). Once the number is in engineering notation, the prefix will always fall in the range: $1 \leq \text{prefix} < 1000$.

It is also quite simple to use power-of-ten notation on a calculator. For example, an entry of 9.87×10^{-6} would look like this: 9.87E-6. The "E" stands for the "10" calculator function and also is commonly seen as "Exp" or "EE." The "–" is a negative sign, not a subtraction sign. Some calculators have buttons with "(–)" or "+ /–." A calculator also reports extremely large or small numbers in scientific notation in the same form used to enter them. For example, when calculating 123^{45} , using the y^x key, a calculator will show something like 1.11e94. The benefits of scientific notation are clear in this problem: Scientific notation saves the user from having to write out the entire number (95 digits!).

MOVIE 1.3 Using Scientific Notation on a Calculator



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Solution to Scenario Question

Solution to the Scenario Question

Since A0 corresponds to a0, A1 to a1, and so forth, we have:

$$
\begin{aligned}
T=&\left(1.978425\times 10^{1}\right)V-\left(2.001204\times 10^{-1}\right)V^{2}\\
&+\left(1.036969\times 10^{-2}\right)V^{3}-\left(2.549687\times 10^{-4}\right)V^{4}\\
&+\left(3.585153\times 10^{-6}\right)V^{5}-\left(5.344285\times 10^{-8}\right)V^{6}\\
&+\left(5.099890\times 10^{-10}\right)V^{7}
\end{aligned}
$$

And a thermocouple measurement of 18.3 mV corresponds to…

$$
\begin{aligned}
T=&\left(1.978425\times 10^{1}\right)(18.3)-\left(2.001204\times 10^{-1}\right)(18.3)^{2}\\
&+\left(1.036969\times 10^{-2}\right)(18.3)^{3}-\left(2.549687\times 10^{-4}\right)(18.3)^{4}\\
&+\left(3.585153\times 10^{-6}\right)(18.3)^{5}-\left(5.344285\times 10^{-8}\right)(18.3)^{6}\\
&+\left(5.099890\times 10^{-10}\right)(18.3)^{7}
\end{aligned}
$$

$T=336°C$

…a temperature of 336 degrees Celsius.

Practice Exercises

Exercise 1

The following numbers are all larger than one. Change each to **scientific notation**—with only one digit remaining to the left of the decimal point in the final answer.

Example: $3860 = 3.86 \times 10^3$

a. 38,600 =

b. 157,300 =

c. 300,000,000 =

d. 147 =

e. 93,000,000 =

Exercise 2

The following numbers are all less than one. Change each to **scientific notation**—with only one digit (other than zero) remaining to the left of the decimal point in the final answer.

Example: $0.015 = 1.5 \times 10^{-2}$

a. 0.0036 =

b. 0.715 =

c. 0.000025 =

d. 0.002 =

e. 0.00083 =

Exercise 3

The following numbers are written in power-of-ten notation. Change each to **decimal notation**.

Examples: $8.36 \times 10^{-1} = 0.836$
$3.01 \times 10^{3} = 3010$

a. $81.5 \times 10^{-1} =$

b. $47.71 \times 10^{-4} =$

c. $326.1 \times 10^{-4} =$

d. $4.771 \times 10^{4} =$

e. $389 \times 10^{-5} =$

f. $3 \times 10^{8} =$

Exercise 4

Change the following numbers to power-of-ten notation by filling in the correct **prefixes**.

Example: $3860 = 3.86 \times 10^{3}$

a. $38{,}600 = \quad \times 10^{2}$

b. $157{,}300 = \quad \times 10^{4}$

c. $23{,}600 = \quad \times 10^{5}$

d. $0.00147 = \quad \times 10^{-3}$

e. $0.056 = \quad \times 10^{-2}$

f. $0.0791 = \quad \times 10^{-3}$

g. Indicate which of the numbers above have been rewritten in scientific notation.

Exercise 5

Multiply or divide the expressions below with a calculator then write the answers in **scientific notation**.

a. $\left(1.25 \times 10^{12}\right) + \left(8.7 \times 10^{11}\right) =$

b. $\left(3.2 \times 10^{12}\right) + \left(8.7 \times 10^{11}\right) =$

c. $26 \div \left(4.6 \times 10^{14}\right) =$

d. $\left(4.9 \times 10^{-32}\right) \times \left(1.6 \times 10^{33}\right) =$

e. $12^{45} =$

Solutions to Practice Exercises

Exercise 1

1. a. 3.86×10^{4}

   b. 1.573×10^{5}

   c. 3×10^{3}

   d. 1.47×10^{2}

   e. 9.3×10^{7}

Exercise 2

2. a. 3.6×10^{-3}

   b. 7.15×10^{-1}

   c. 2.5×10^{-5}

   d. 2×10^{-3}

   e. 8.3×10^{-4}

Exercise 3

3.
  a. 8.15
  b. 0.004771
  c. 0.03261
  d. 47710
  e. 0.00389
  f. 300,000,000

Exercise 4

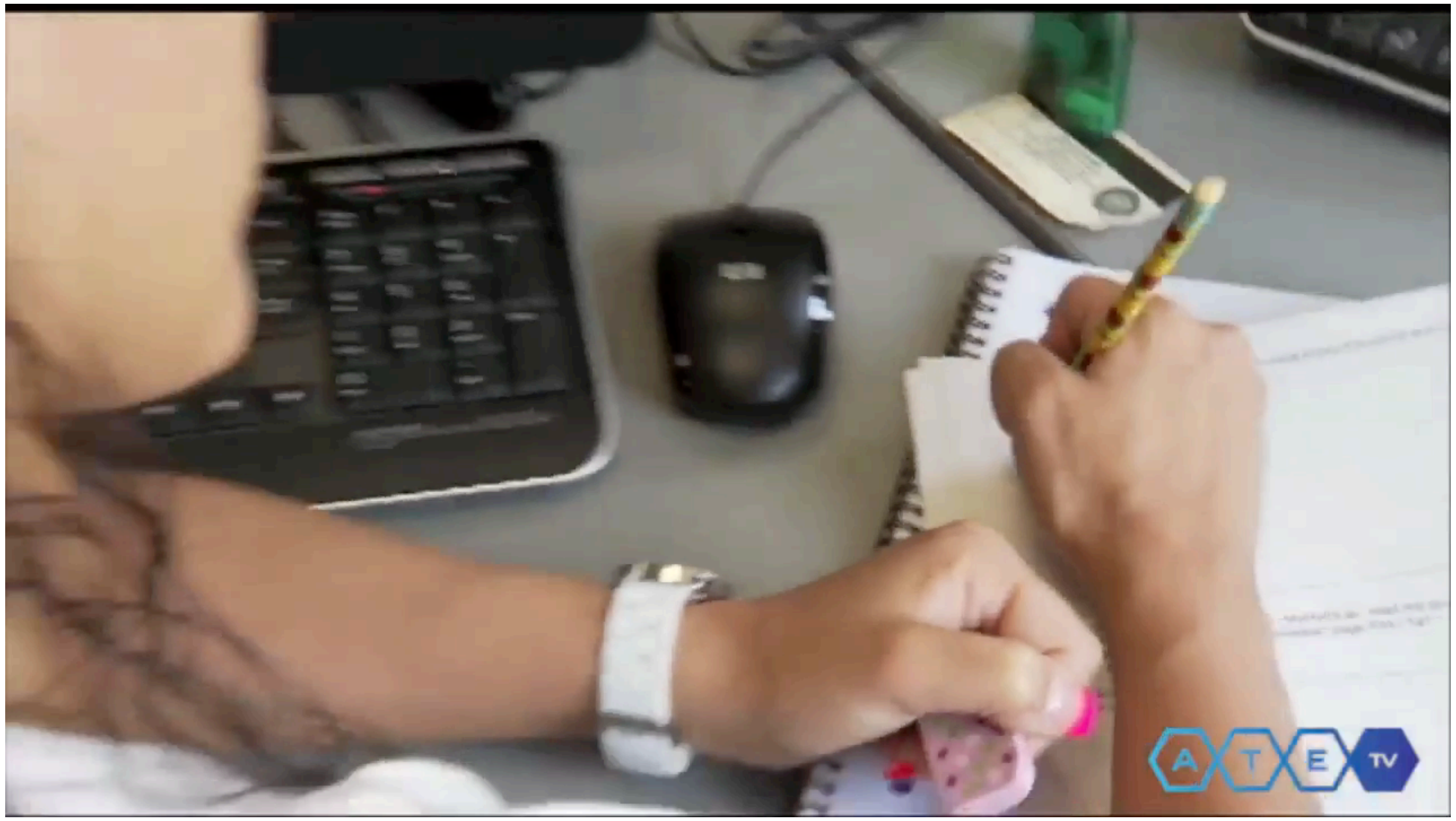
4.
  a. 386
  b. 15.73
  c. 0.236
  d. 1.47
  e. 5.6
  f. 79.1
  g. d and e

Exercise 5

5.
  a. 2.12×10^{12}
  b. 1.89×10^{11}
  c. 5.65×10^{-14}
  d. 7.84×10^{1}
  e. 3.66×10^{48}

Career Video

Mathematics At Work



To be successful in any technical career, the experts all agree that a foundation of strong mathematical skills is critical.

Unit Conversion

2

Objectives

When you have completed this section, you should be able to do the following:

1. Multiply and divide numbers that contain units and obtain answers that correctly express numbers and units
2. Express numerical values and units (such as 3.4×10^{-3} mm) in prefix notation (3.4 millimeters)
3. Given an equation with several terms, check the dimensions for each item and then check the equation as a whole
4. Convert a physical quantity expressed in one set of units to an equivalent

Scenario

Your food processing plant uses tubular heat exchangers to heat and sterilize various liquids prior to packaging. A new tubular heat exchanger is advertised to have an overall heat transfer coefficient of $1950 \frac{W}{m^2 C°}$. The specifications for your plant require exchangers that have coefficients greater than $300 \frac{Btu}{h \cdot ft^2 \cdot F°}$. Will the coefficient for this new heat exchanger satisfy the plant specifications?

Before looking at the solution, work through the lesson to further develop your skills in this area.

The Basics of Units

Part 1

Most quantities used and measured by technicians are described by numbers that include units. For example, consider the following:

a. 10 feet: 10 is the number; feet is the unit.

b. 20 miles/hour: 20 is the number; miles/hour is the unit.

c. 3cm^3 : 3 is the number; cm^3 is the unit.

d. $5\,\text{lb/ft}^2$: 5 is the number; lb/ft^2 is the unit.

When numbers that contain units are multiplied or divided, the numbers and units are handled separately. Study the following examples:

a. $10\text{ ft} \times 20\text{ ft} = (10 \times 20) \times (\text{ft} \times \text{ft}) = 200\text{ ft}^2$.
(Multiplying $\text{ft} \times \text{ft}$ is feet squared, written as ft^2 .)

b. $\left(20\,\frac{\text{mi}}{\text{h}}\right) \times 10\text{ h} = (20 \times 10) \times \left(\frac{\text{mi}}{\cancel{\text{h}}} \times \cancel{\text{h}}\right) = 200\text{ mi}$

(Multiplying $\frac{\text{mi}}{\text{h}} \times \text{h}$ is simplified by canceling the "h" that occurs in both the numerator and the denominator.)

c. $\left(5\,\frac{\text{lb}}{\text{ft}^2}\right) \times 10\text{ ft}^2 = (5 \times 10) \times \left(\frac{\text{lb}}{\cancel{\text{ft}^2}} \times \cancel{\text{ft}^2}\right) = 50\text{ lb}$

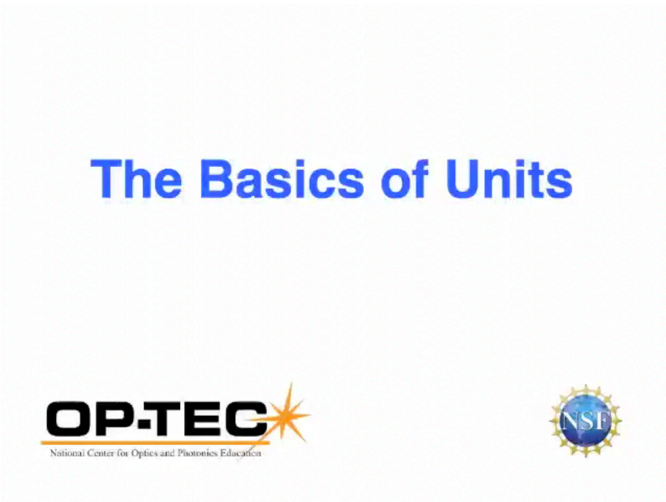
(The " ft^2 " in the numerator cancels " ft^2 " in the denominator.)

d. $5\,g \div 10\,cm^2 = \dfrac{5\,g}{10\,cm^2} = \left(\dfrac{5}{10}\right) \times \left(\dfrac{g}{cm^2}\right) = 0.5\,g/cm^2$

e. $20\ N \div 5\ m^2 = \dfrac{20\ N}{5\ m^2} = \left(\dfrac{20}{5}\right) \times \left(\dfrac{N}{m^2}\right) = 4\ N/m^2$

f. $5\ \dfrac{lb}{ft^3} \times 2\ ft = (5 \times 2) \times \left(\dfrac{lb \times ft}{ft^3}\right) = 10\left(\dfrac{lb \times \cancel{ft}}{\cancel{ft} \times ft \times ft}\right) = 10\ lb/ft^2$

MOVIE 2.1 The Basics of Units



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Part 2

The prefixes given in Table 2.1 are defined for numbers given in powers of ten. In the table, the most important prefixes are indicated by asterisks(*). Begin by learning the most important ones. Then review the others often until you recognize them.

Example: Use Table 2.1 to see that the SI (International System of Units) prefixes and units for the numbers given below are correct.

TABLE 2.1 SI Prefixes

Factor by Which the Unit Is Multiplied	Prefix	Symbol
10^{12}	tera	T
*10^{9}	giga	G
*10^{6}	mega	M
*10^{3}	kilo	k
10^{2}	hecto	h
$10^{1} = 10$	deca	da
10^{-1}	deci	d
*10^{-2}	centi	c
*10^{-3}	milli	m
*10^{-6}	micro	µ
*10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

* Most commonly used

100×10^{-2} meters = 100 <u>centi</u> meters (cm)

1×10^{3} meters = 1 <u>kilo</u> meter (km)

5×10^{-3} meters = 5 <u>milli</u> meters (mm)

1.8×10^{-6} seconds = 1.8 <u>micro</u>seconds (s)

2×10^{12} watts = 2 <u>tera</u> watts (TW)

1×10^{3} cal = 1 <u>kilo</u> calories (kcal)

0.2×10^{6} newtons = 0.2 <u>mega</u> newtons (MN)

Part 3

Operations performed with units are similar to operations performed with numbers. Therefore, if the units that accompany a numerical answer aren't correct, an error has been made in the problem-solving process. To make this point more clear, look at the following equation:

$$S = vt + \frac{1}{2} at^2$$

where	S	=	Distance
	v	=	Velocity (distance/time)
	t	=	Time
	a	=	Acceleration (distance/ time^2)

This equation involves solving for distance (S). Therefore, each term, such as " vt " and " $\frac{1}{2}at^2$ " on the right side of the equation, must also have units of "distance." Let's show this by first substituting the correct units for v, t, and a on the right side, then canceling units where appropriate.

$S = vt + \frac{1}{2} at^2$ (Ignore constants, such as $\frac{1}{2}$, that carry no units.)

$S = \left(\frac{\text{distance}}{\cancel{\text{time}}} \times \cancel{\text{time}}\right) + \left(\frac{\text{distance}}{\cancel{\text{time}^2}} \times \cancel{\text{time}^2}\right)$ (Cancel time units.)

S = distance + distance

S = distance

The equation is correct. That's because a **distance** plus a **distance** does equal a distance, and we know that S should be a distance unit.

Now let's substitute SI units in the equation and show the same thing. Use *t* in seconds, *v* in meters/s, and *a* in meters/s^2 . Then:

$S = vt + \frac{1}{2} at^2$

$S = \left(\frac{\text{meters}}{\cancel{s}} \times \cancel{s}\right) + \left(\frac{\text{meters}}{\cancel{s^2}} \times \cancel{s^2}\right)$ (Cancel s units.)

S = meters + meters

S = meters

The equation is dimensionally correct. Since the left side S is a distance, it should have "distance" units. It does; the right side is in meters (a distance unit).

Part 4

Just as apples and oranges don't add up, neither do numbers that carry different units. For example, "2 inches + 1 foot" doesn't equal either 3 inches or 3 feet. To add 2 inches to 1 foot, you must **first** express both terms in the **same** units.

Thus, if you change 1 foot to 12 inches and then add "2 inches + 12 inches," you get a correct answer: 14 inches. So, "2 inches + 1 foot" = 14 inches. If you change 2 inches to $\frac{1}{6}$ foot and then add "$\frac{1}{6}$ft + 1ft ," you will get another correct answer: $1\frac{1}{6}$ ft.

Suppose you are asked to add the following:

1 meter + 40 centimeters =

Since the units aren't alike, you can't add the two numbers as they stand. But you can change 1 meter to 100 centimeters and then add.

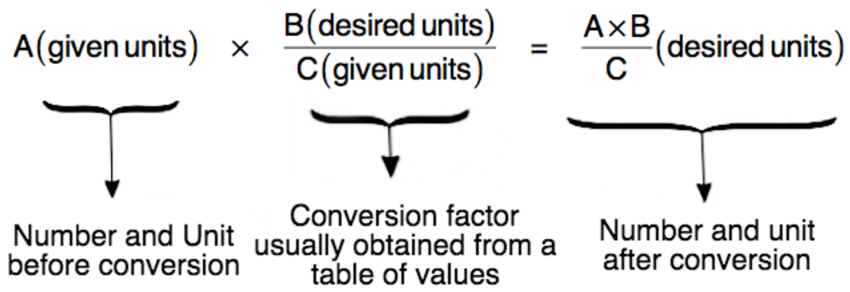
100 cm + 40 cm = 140 cm (a correct answer)

Or, you can change 40 cm to 0.40 meter (since 40 cm equals 40/100 of a meter, and 40/100 = 0.40) and then add.

1 m + 0.40 m = 1.40 m (also a correct answer)

To convert units from one form to another, such as feet to inches or kilograms to grams, follow the general conversion process shown in Figure 2.1. The letters A, B, and C represent numbers.

FIGURE 2.1 Unit Conversion Process



Let's try some examples that show how the conversion process outlined in Figure 2.1 is applied.

Example 1

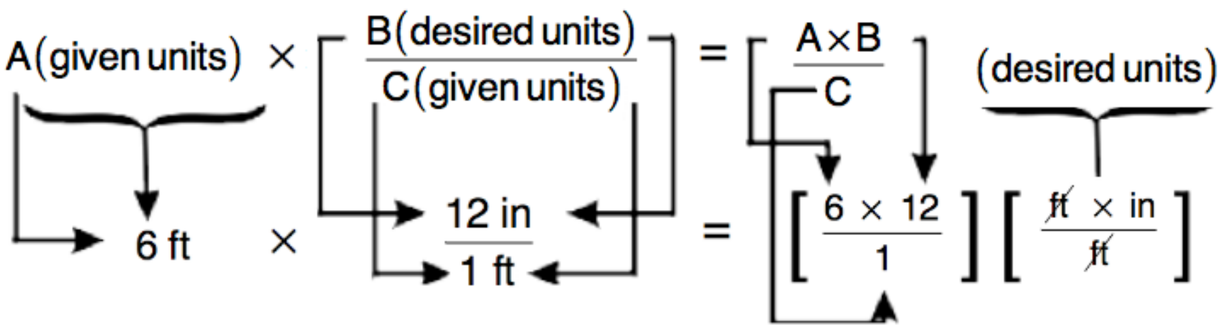
Given: A length of 6 feet

Find: The same length in inches

Solution: From a table of conversion values, we find that 1 ft = 12 in. Therefore, we can form the conversion factor $\frac{\text{B (desired units)}}{\text{C (given units)}}$ as follows:

$$\frac{12 \text{ in}}{1 \text{ ft}}$$

The ratio is equal to 1 because the numerator and denominator are equal lengths. (**Note:** Multiplying something by 1 doesn't change it.) Now follow the process shown in Figure 2.1.



Thus,

$$6 \text{ ft} \times \frac{12 \text{ in}}{1 \text{ ft}} = \left(\frac{6 \times 12}{1}\right)\left(\frac{\cancel{\text{ft}} \times \text{in}}{\cancel{\text{ft}}}\right) = 72 \text{ in}$$

Thus, the result is 6 ft = 72 in.

Note: In the conversion process of Example 1, we multiplied 6 ft by the ratio $\frac{12 \text{ in}}{1 \text{ ft}}$. The ratio is equal to 1. Thus, the length of 6 ft wasn't changed, since it was multiplied by 1. It was only converted to an equivalent length expressed in inches—72 inches.

Example 2

Given: A distance of 10 miles

Find: The same distance in feet

Solution: From a conversion table for lengths, we find that:

$$1 \text{ mile} = 5280 \text{ feet}$$

With this equality we can form two conversion ratios, each equal to 1.

$$\frac{1 \text{ mile}}{5280 \text{ ft}} \text{ and } \frac{5280 \text{ ft}}{1 \text{ mile}} \Rightarrow \frac{\text{B (desired units)}}{\text{C (given units)}}$$

Since we're converting from miles (given units) to feet (desired units), we'll use the ratio $\frac{5280 \text{ ft}}{1 \text{ mile}}$ for the conversion factor. Then, using the process shown in Figure 1, multiply and cancel units.

$$10 \text{ mi} \times \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) = \left(\frac{10 \times 5280}{1}\right)\left(\frac{\cancel{\text{mi}} \cdot \text{ft}}{\cancel{\text{mi}}}\right) = 52{,}800 \text{ ft}$$

Therefore, 10 miles = 52,800 feet.

Example 3

Given: A distance of 52,800 ft

Find: The same distance in inches

Solution: From a conversion table, we find that 1 foot = 12 inches. Since we're changing feet (given units) to inches (desired units), we use the conversion ratio $\frac{12 \text{ in}}{1 \text{ ft}}$. Multiply and cancel units.

$$52{,}800 \text{ ft} \times \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \left(\frac{52{,}800 \times 12}{1}\right)\left(\frac{\cancel{\text{ft}} \bullet \text{in}}{\cancel{\text{ft}}}\right) = 633{,}600 \text{ in}$$

Therefore, 52,800 ft = 633,600 in.

If we combine the results in Examples 2 and 3, we find that 10 miles = 633,600 inches. We found this in two steps. We could have done it in one step. See Example 4.

Example 4

Given: A distance of 10 miles

Find: The same distance in inches

Solution: From a table of conversions between lengths, we are given:

1 mi = 5280 ft and 1 ft = 12 in.

We form the conversion ratios $\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)$ and $\left(\frac{12 \text{ in}}{1 \text{ ft}}\right)$.

Then multiply and cancel units.

$$10 \text{ mi} \times \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) \times \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \left(\frac{10 \times 5280 \times 12}{1 \times 1}\right)\left(\frac{\cancel{\text{mi}} \bullet \cancel{\text{ft}} \bullet \text{in}}{\cancel{\text{mi}} \bullet \cancel{\text{ft}}}\right)$$

Therefore, 10 mi = 633,600 in.

The answer is the one we found in two steps in Examples 2 and 3.

Solution to Scenario Question

Solution to the Scenario Question

You need to convert the metric coefficient to English units (or vice versa). To do that, you can convert watts to the English unit of Btu/h, convert m^2 to ft^2 ; and convert C° to F°. From a table of conversions (e.g., the Appendix of this book):

- 1 watt = 3.413 Btu/h
- 1 meter = 3.281 ft
- 1 C° = 1.8 F°

So, we form and multiply the given value by *unit ratios* so that the given units "cancel," leaving us with the desired units.

$$1950\frac{\cancel{W}}{\cancel{m^2}\cancel{C°}} \times \frac{3.413\ \text{Btu/h}}{1\ \cancel{W}} \times \left(\frac{1\ \cancel{m}}{3.281\ \text{ft}}\right)^2 \times \frac{1\ \cancel{C°}}{1.8\ \text{F°}} = 343.5\frac{\text{Btu}}{\text{h} \bullet \text{ft}^2 \bullet \text{F°}}$$

Note: Multiply by the meter-to-foot conversion twice, that is, squared, to cancel the " m^2 " unit.

So, the new exchanger's coefficient of $343.5\frac{\text{Btu}}{\text{h} \cdot \text{ft}^2 \cdot \text{F°}}$ is indeed greater than the specification minimum of $300\frac{\text{Btu}}{\text{h} \cdot \text{ft}^2 \cdot \text{F°}}$.

Practice Exercises

Multiply or divide the quantities given below as indicated, and obtain each answer as a correct number and unit.

Exercise 1

a. $5\ \text{lb} \times 10\ \text{ft} =$

b. $10\ \text{lb} \times 2\ \text{ft}^2 =$

c. $100\ \text{N} \div 5\ \text{m}^2 =$

d. $1000\ \text{cm}^3 \times 13.6\ \dfrac{\text{g}}{\text{cm}^3} =$

e. $16\ \dfrac{\text{miles}}{\text{hour}} \times 0.5\ \text{hour} =$

f. $5\ \text{hours} \times 60\ \dfrac{\text{minutes}}{\text{hour}} =$

g. $62.4\ \dfrac{\text{lb}}{\text{ft}^3} \times 10\ \text{ft} =$

h. $10\ \dfrac{\text{N}}{\text{m}^3} \times\ 2\ \text{m} =$

Exercise 2

Place the names and symbols for the following factors in the spaces provided. The answer to the first one is provided.

Factor	Name	Symbol
10^{3}	kilo	k
10^{-3}		
10^{6}		
10^{-6}		
10^{-2}		
10^{9}		
10^{-9}		
10^{-12}		

Check answers by comparing to Table 2.1.

Exercise 3

For each value given below, change the power-of-ten notation to a number. Then add the correct prefix symbol to the unit.

5.6×10^{3} meters = 5.6 kilometers (km)

6.8×10^{3} calories = ____________

10×10^{-2} meters = ____________

5×10^{-3} meters = ____________

3.14×10^{-9} seconds = ____________

Exercise 4

Use Table 2.1 to identify the correct prefixes for the power-of-ten numbers given below.

10^{-3} meters = ____________

10^{-3} grams = ____________

10^{-6} calories = ____________

10^{-2} meters = ____________

Exercise 5

Rewrite “15,600 grams” using power-of-ten notation and a unit prefix name that involves kilograms.

Exercise 6

Given the equation:

$$v_f = v_i + at$$

where v_f = Speed (ft/s or m/s)
v_i = Initial speed (ft/s or m/s)
a = Acceleration (ft/s^2 or m/s^2)
t = Time (s)

Substitute the proper units for v_i, a, and t, first in the English system and then in SI. Show that each term has dimensions of ft/s or m/s, hence speed—the correct units for v_f.

Exercise 7

Given the equation:

$$R_E = \frac{\rho L}{A}$$

where R_E = Electrical resistance in ohms (Ω)
ρ = Electrical resistivity in ohm cm
L = Length in cm
A = Cross-sectional area in cm^2

Substitute the proper units for ρ, L, and A into the equation. Show that the terms on the right reduce to ohms (Ω). This is the correct unit for the resistance R_E on the left side.

Exercise 8

Convert 150 centimeters to an equal length in millimeters. A table of conversions gives 1 cm = 10 mm. This solution is partially set up as follows:

$$150 \text{ cm} \times \frac{10 \text{ mm}}{1 \text{ cm}} =$$

Exercise 9

Convert 21 liters to an equal volume in gallons. A table of conversions gives 1 gal = 3.785 liters.

Exercise 10

Convert 40 centimeters to an equal length in meters. A table of conversions gives 1 m = 100 cm.

Exercise 11

Convert 5 hours to an equal time in seconds. A table of conversions gives 1 h = 3600 s.

Exercise 12

Convert 10.5 mi/h to an equal speed in ft/s. A table of conversions gives 1 mi = 5280 ft and 1 h = 3600 s. Solve this problem in one step, as was done in Example 4.

Solutions to Practice Exercises

Solutions to Practice Exercises

1. a. 50 lb • ft

   b. 5 lb/ft^2

   c. 20 N/m^2

   d. 13,600 g

   e. 8 miles

   f. 300 minutes

   g. 624 lb/ft^2

   h. 20 N/m^2

2. See Table 2.1.

3. 6.8 kilocalories (kcal)
   10 centimeters (cm)
   5 millimeters (mm)
   3.14 nanoseconds (ns)

4. millimeters (mm)
   milligrams (mg)
   microcalories (μcal)
   centimeters (cm)

5. 1.56×10^1 kg

6. $v_f = v_i + at$ $\quad$ $v_f = v_i + at$

$$\frac{ft}{s} = \frac{ft}{s} + \left(\frac{ft}{s^{\cancel{2}}}\right)(\cancel{s}) \qquad \frac{m}{s} = \frac{m}{s} + \left(\frac{m}{s^{\cancel{2}}}\right)(\cancel{s})$$

$$\frac{ft}{s} = \frac{ft}{s} \qquad \frac{m}{s} = \frac{m}{s}$$

7. $R_E = \dfrac{\rho \ell}{A}$

$$\Omega = \frac{(\Omega \cdot \cancel{cm})\cancel{cm}}{\cancel{cm^2}}$$

$$\Omega = \Omega$$

8. $150\ \cancel{cm} \times \dfrac{10\ mm}{1\ \cancel{cm}} = 1500\ mm$

9. $21\ \cancel{L} \times \dfrac{1\ gal}{3.785\ \cancel{L}} = 5.55\ L$

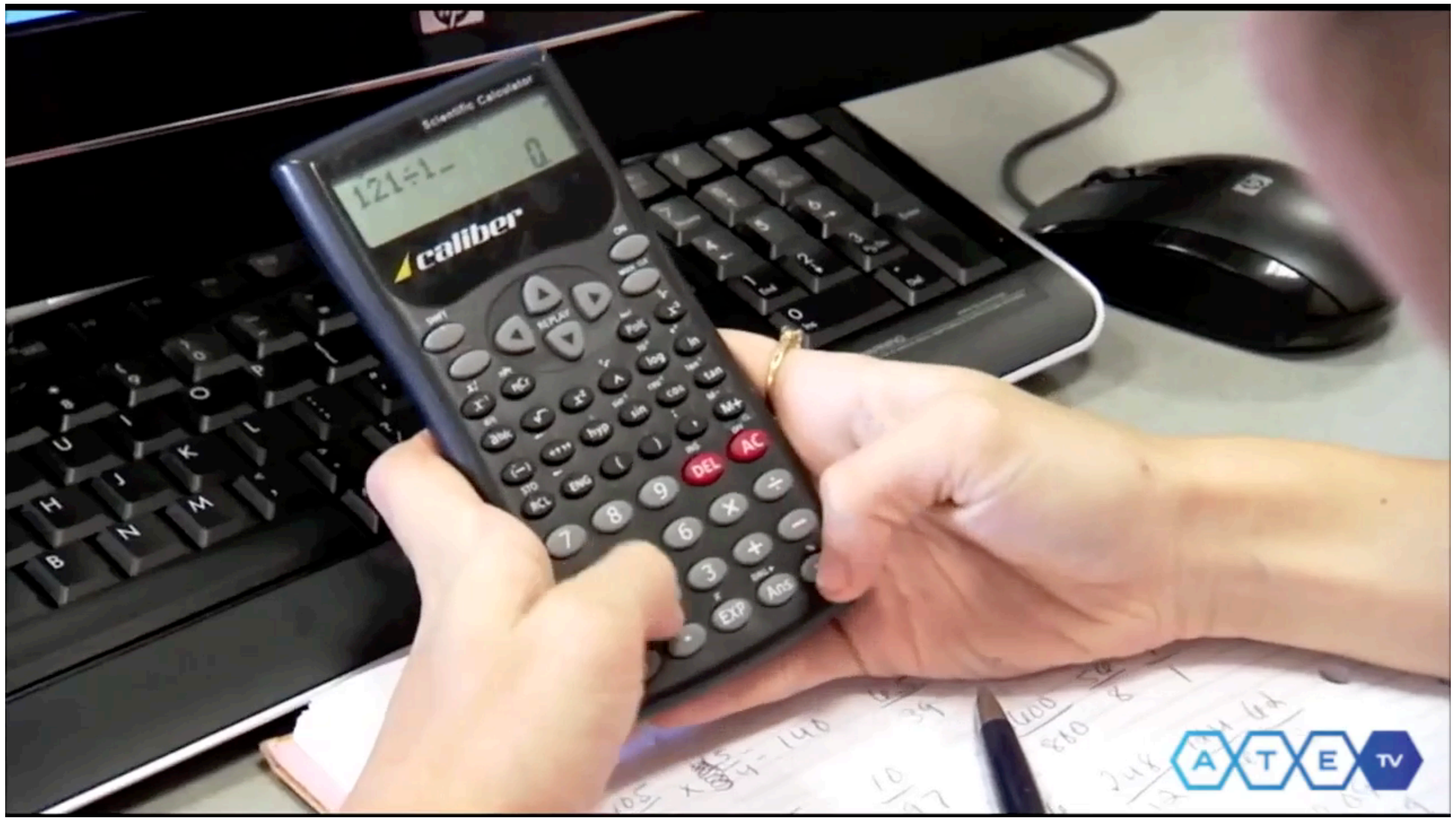
10. $40\ \cancel{cm} \times \dfrac{1\ m}{100\ \cancel{cm}} = 0.40\ m$

11. $5\ \cancel{h} \times \dfrac{3600\ s}{1\ \cancel{h}} = 18{,}000\ s$

12. $\dfrac{10.5\ mi}{\cancel{h}} \times \dfrac{5280\ ft}{1\ \cancel{mi}} \times \dfrac{1\ \cancel{h}}{3600\ s} \times = 15.4\ ft/s$

Career Video

Math Is More Than Just Numbers



Feeling stumped by math? Visit a college prep class where students are getting up to speed on basic skills they need to move ahead.

Introductory Algebra

3

Objectives

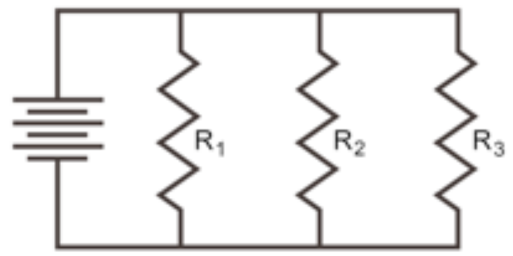
When you complete this lesson, you should be able to do the following:

1. Simplify an expression (order of operations)
2. Rearrange formulas so that one variable is isolated
3. Substitute into formulas and solve for unknown quantities

Scenario

Resistors wired in parallel across a voltage present a total resistance R_T calculated by the formula:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + L$$



If R_1 = 33 ohms and R_2 = 28 ohms, what must R_3 be so that R_T = 20 ohms?

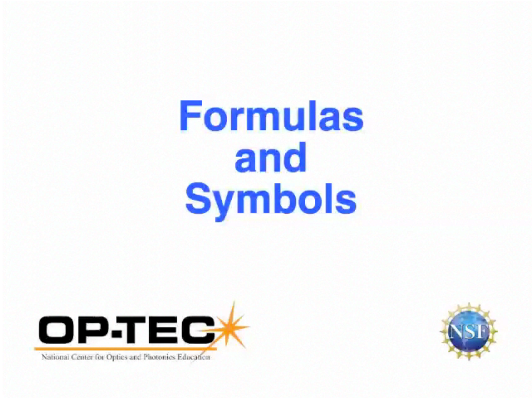
Before looking at the solution, work through the lesson to further develop your skills in this area.

Formulas

What is a formula?

In mathematics a formula is a way of using symbols to write a sentence. The following examples show formulas followed by the complete sentences they represent.

MOVIE 3.1 Formulas and Symbols



Click or Tap to Watch

Formula: $p = 2l + 2w$

Sentence: The perimeter of a rectangle is the sum of twice the length and twice the width.

Formula: $i = prt$

Sentence: The simple interest is the product of the principal, the annual rate, and the time in years.

Which do you think is easier to read and remember—the formula (written in mathematics symbols) or the sentence (written in words)?

What does a formula do? A formula shows how some quantities relate to each other. For example, $C = \pi d$ shows that the circumference of a circle is always equal to the diameter of the circle multiplied by the number π (approximately 3.1416). Another way to express this relationship is to say, " The distance *around* a circle is always roughly three times the distance *across* the circle."

A formula is often the shortest, easiest way to write the relationship between two or more quantities. Once a formula is written correctly, it is ready to be used. Always take care to substitute correct numbers and units for known symbols in an equation. Only then can you calculate the number and unit of the unknown (isolated) symbol correctly.

Order of Operations

It is important to remember a particular order of operations when simplifying the expressions on either side of the equal sign. The mathematical operations are *addition, subtraction, multiplication,* and *division*. A combination of numbers and operations, such as $130 \div 10 + 3 \times 2$, is called a numerical expression. An algebraic expression contains one or more variables. For example, $2x^2 - 5x + 3$ is an algebraic expression.

To evaluate an expression means to find its value. When you evaluate an algebraic expression you replace the variable with a value. Rules must be followed so that everyone knows which operation to perform first and to ensure that everyone finds the same answer when evaluating an expression. The rules are called the **order of operations**.

1. **P**arentheses—Evaluate all operations inside parentheses and brackets (grouping symbols).

2. **E**xponents—Evaluate all exponents and powers.

3. **M**ultiplication and **D**ivision—Multiply and divide from left to right.

4. **A**ddition and **S**ubtraction—Add and subtract from left to right.

Movie 3.2 Order of Operations

Click or Tap to Watch

Helpful reminder: **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally (PEMDAS)

Example 1

Simplify: $8 - 3(5 + 2^4)$

Solution:	$8 - 3(5 + 16)$	Inside parenthesis, do exponent first
	$8 - 3(21)$	Next finish parentheses with addition
	$8 - 63$	Then multiplication
	-55	Subtraction last

Example 2

Given: The formula $V = I \times R$

$V = 10$ volts (V) and $R = 2$ ohms (Ω).

Find: Current, I

Solution: The equation isn't in the form with I isolated. So isolate I.

$$V = I \times R$$

$$\frac{V}{R} = \frac{I \times \cancel{R}}{\cancel{R}}$$ (Divide both sides by R, then cancel R's.)

$$\frac{V}{R} = I$$

$$I = \frac{V}{R}$$ (Reverse the order of the equation.)

Now, with I isolated, substitute in the given values of V and R.

$$I = \frac{10\ V}{2\ \Omega} = \left(\frac{10}{2}\right) \times \left(\frac{V}{\Omega}\right) = 5\left(\frac{V}{\Omega}\right) \quad \left(1\frac{V}{\Omega} \equiv 1\ A\right)$$

$$I = 5\ A$$

The current is 5 amperes.

Example 3

Rearrange the formula for the volume of a sphere so that you can isolate and solve for the radius *r*.

$$V=4/3\pi r^3$$

First, ask yourself this question: "What must I do to each side of the equation to free the term r^3 ?"

In this formula, r^3 is multiplied by 4, multiplied by , and divided by 3. So, if you divide by 4, divide by , and multiply by 3, you will have only r^3 left. Let's go through the steps carefully.

Notice that r^3 becomes isolated in the product $\left(\frac{4}{3}\pi r^3\right)$ if you do the following:

- Divide by 4
- Divide by π
- Multiply by 3

But dividing by 4, dividing by π , and multiplying by 3 is the same as multiplying by $\frac{3}{4\pi}$, all at once. Apply this same operation to *each* side of the equation, as follows.

$$\left(\frac{3}{4\pi}\right) \times V = \left(\frac{\cancel{3}}{\cancel{4\pi}}\right)\left(\frac{\cancel{4\pi}}{\cancel{3}}r^3\right)$$

The terms on the right simplify to r^3 , since $\frac{3}{4\pi} \times \frac{4\pi}{3} = 1$. The formula that has been rearranged becomes:

$$\frac{3V}{4\pi}=r^3$$

You can rewrite this equation with r^3 on the left.

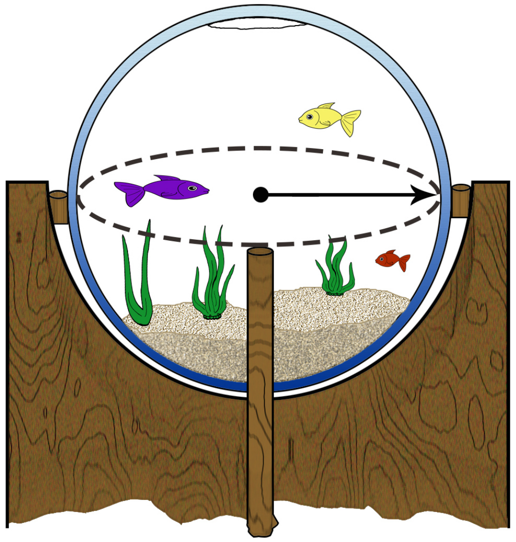
$$r^3=\frac{3V}{4\pi}$$

This is the formula that helps you find the radius of a sphere if you know the volume of the sphere.

For a given problem, if you know the volume V, you would use your calculator to enter the expression $\frac{3V}{4\pi}$ and then use the cube root key or the $\sqrt[x]{y}$ key to get the answer for the radius r. Let's try that in the following example.

Example 4

A spherical tank must hold 6 cubic feet of liquid. What is the tank's radius to the nearest tenth of a foot?



Rewrite the formula, substituting the number values that you know for the variables.

$$r=\sqrt[3]{\frac{3V}{4\pi}}$$

$$r=\sqrt[3]{\frac{3(6\text{ ft}^3)}{4\pi}}$$

Now, solve for r with the help of your calculator. You should find that r = 1.1 ft (rounded).

If you didn't get the answer r = 1.1 ft, try again. If you still don't, you can check your use of the calculator against the following steps. (Remember, there's always more than one way to get the answer. Your method may be better and shorter than the one outlined below.)

One way to solve for r in the formula $r=\sqrt[3]{\frac{3(6\text{ ft}^3)}{4\pi}}$:

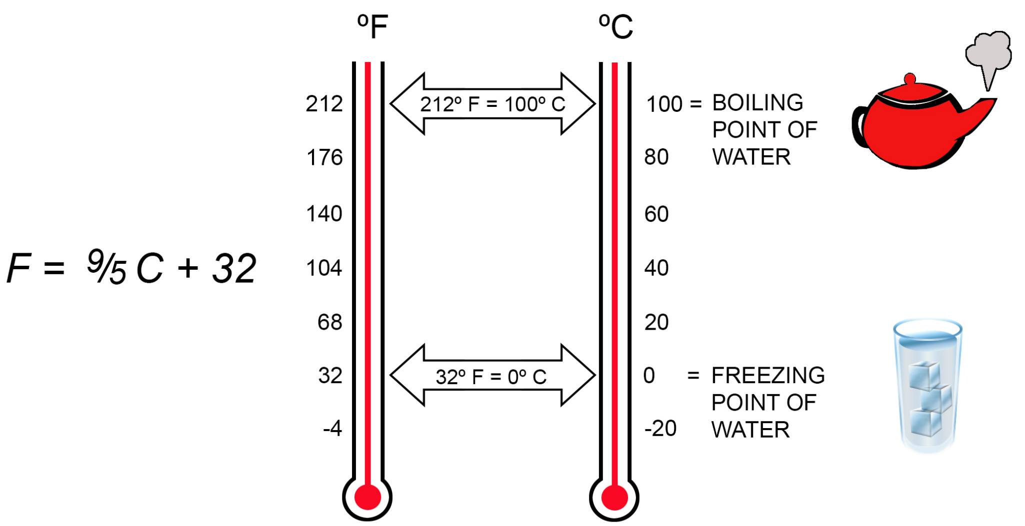
Clear the calculator
Enter 3
Press ×
Enter 6
Press ÷
Enter 4
Press ÷
Press the π key
Press = (display should show 1.4323945)
Press INV or 2nd Function key
Press y^x key
Press 3 (for cube root)
Press = (not needed on some calculators)
Answer 1.1272517 appears in window.
Round to the nearest tenth to get r = 1.1 ft.

Thus, to have a volume of 6ft^3, you've found that a sphere must have a radius of about 1.1 ft.

If you want to check that answer, put r = 1.1 ft back in the original formula for the volume of a sphere, $V = 4/3r^3$. Should you expect to get $V = 4/3\pi r^3$ exactly? Why? What value should you use for r to get $V = 6ft^3$ as closely as possible?

Now let's look at the formula that shows the relationship between the Fahrenheit and Celsius temperature scales, as illustrated in the figure below.

FIGURE 3.1 Fahrenheit and Celsius Comparison



The formula to convert a temperature on the Celsius scale (°C) to the equivalent temperature on the Fahrenheit scale (°F) looks like this:

$$F = 9/5C + 32$$

Notice that the constant 32 is added to the expression 9/5C.

Example 5

Rewrite this formula so you can use it to change a Fahrenheit temperature to one on the Celsius scale. Begin with the given formula:

$$F = 9/5C + 32$$

(**Note:** F, C, and 32 are generally written as °F, °C, and 32°—with the degree symbol attached. We have, however, omitted the degree symbol to keep the formula as simple as possible.)

$$F = 9/5C + 32$$

Since "32" is added in the equation, *subtract* 32 from each side to free the term 9/5C —the term that contains C:

$$F - 32 = 9/5C + 32 - 32$$

After you simplify the right side, you get:

$$F - 32 = 9/5C$$

Now multiply the right side by 5 and divide by 9 to "free" the C of the coefficient 9/5 . That's the same as multiplying by the fraction 5/9 . Do the same thing to the left side.

$$\frac{5}{9}(F - 32) = \frac{\cancel{5}}{\cancel{9}}\left(\frac{\cancel{9}}{\cancel{5}}C\right)$$

Notice that the multiplying fraction 5/9 *must multiply the entire left side and the entire right side*. So put parentheses around each side before you multiply.

After you divide the fives and nines on the right side, you have this equation:

$$5/9(F - 32) = C$$

or

$$C = 5/9(F - 32)$$

Remember, a number or letter written next to parentheses means that you multiply what is in the parentheses by that number or letter.

Thus, the expression $5/9(F - 32)$ means that you calculate $(F - 32)$ and multiply that result by 5/9.

You have learned how to begin with a formula such as $F = 9/5C + 32$ and rearrange it to isolate the variable C—to get the formula $C = 5/9(F - 32)$. You can use the "rearranged" formula if you know the Fahrenheit temperature F and you want to calculate the corresponding Celsius temperature C.

Example 6

Find what 36°F is on the Celsius scale.

First, choose the version of the formula that has the variable you are looking for isolated on one side of the equal sign.

Since you are looking for a temperature in °C, you would choose the second version of the formula, with C isolated on the left side:

$$C = 5/9(F - 32)$$

Substitute the number values from the problem for the variables you know in the formula. You are given that the Fahrenheit temperature is 36°. Therefore,

$$C = 5/9(36° - 32°)$$

Use your calculator to solve this formula for C. If you round your answer to two decimal places and attach the proper units, you should get C = 2.22°C. How did you do? Did you use the parenthesis keys? Can you solve for C without using the parenthesis keys?

Some problems require more algebraic steps to come to solutions. For example, solve for q in the following equation:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

where f = Focal length of the lens
p = Distance of the object being viewed from the center of the lens
q = Distance of the image from the center of the lens

Example 7

Given: The focal length of the lens (f = 12 mm) and the distance from the object to the lens (p = 15 cm)

Find: The distance from the image to the lens (q)

Solution: Rearrange the equation so that q is isolated. Then substitute in values of other variables and solve for q.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{p} = \frac{1}{p} + \frac{1}{q} - \frac{1}{p}$$ (Subtract from both sides.)

$$\frac{1}{f} - \frac{1}{p} = \frac{1}{q}$$ (Subtract out ’s from right side.)

$$\frac{p}{fp} - \frac{f}{fp} = \frac{1}{q}$$ (Get common denominator on left side.)

$$\frac{p - f}{fp} = \frac{1}{q}$$ (Combine terms on left side.)

We are not there yet; q still must be isolated in the numerator.

$$q\left(\frac{p - f}{fp}\right) = \frac{1}{\cancel{q}} \times \cancel{q} \qquad \text{(Multiply both sides by } q.\text{)}$$

$$q\left(\frac{\cancel{p - f}}{\cancel{fp}}\right) \times \left(\frac{\cancel{fp}}{\cancel{p - f}}\right) = 1 \times \left(\frac{fp}{p - f}\right)$$

(Multiply both sides by the reciprocal.)

$$* \; q = \frac{fp}{p - f}$$

Before substituting in values of the given variables, make sure all units work together.

$$12 \; \cancel{mm} \times \frac{1 \text{ cm}}{10 \; \cancel{mm}} = 1.2 \text{ cm}$$

We were given *cm* and *mm*, but we know how to convert *mm* to *cm*.

$$q = \frac{(1.2 \text{ cm})(15 \text{ cm})}{(15 \text{ cm} - 1.2 \text{ cm})} \qquad \text{Substitute the values of variables.}$$

$$q = 1.3 \text{ cm(rounded)} \qquad \text{Simplify.}$$

Therefore, the image is 1.3 cm from the lens.

Before we move on, it is important to note the significance of the derived formula (*). This formula will always find the distance from the image to the lens given the focal length of the lens and the distance from the object to the lens. In general, any formula of the form

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{C}$$

can be rearranged as $C = \frac{AB}{B - A}$

or solving for B as $B = \frac{AC}{C - A}$

These are useful when finding equivalent capacitance (C_e) of two capacitors (C_1 and C_2) in series because:

$$\frac{1}{C_e} = \frac{1}{C_1} + \frac{1}{C_2}$$

can now be written $C_1 = \frac{C_eC_2}{C_2 - C_e}$ or $C_2 = \frac{C_eC_1}{C_1 - C_e}$

You may encounter an even more complex equation like this one:

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}$$

If you were asked to solve for s_2 on this problem, the algebra would get quite tedious. So it would be easier to plug in numbers for the given values and then solve for s_2 . The next example will demonstrate this idea.

Example 8

Given: A plastic rod (n_2 = 1.48) in air (n_1 = 1), with one end ground to a convex spherical shape of radius 4 cm (R = 4 cm)

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}$$

Find: The position of the image (s_2) of a small object 20 cm (s_1 = 20 cm) from the spherical end and on axis

Solution: Substitute in given values then solve for s_2 .

$$\frac{1}{20} + \frac{1.48}{s_2} = \frac{1.48 - 1}{4}$$ (Substitute in the given values.)

$$\frac{1}{20} - \frac{1}{20} + \frac{1.48}{s_2} = \frac{0.48}{4} - \frac{1}{20}$$ (Subtract from each side.)

$$\frac{1.48}{s_2} = 0.07$$ (Simplify right side with calculator.)

$$\cancel{s_2} \times \frac{1.48}{\cancel{s_2}} = 0.07 \times s_2$$ (Multiply both sides by s_2 .)

$$1.48 = 0.07 \times s_2$$

$$\frac{1.48}{0.07} = \frac{\cancel{0.07} \times s_2}{\cancel{0.07}}$$ (Divide both sides by 0.07.)

$$21.1 = s_2$$ (Simplify both sides.)

This solution turns out to be fairly simple. The derivation of the formula for s_2 would have been more difficult than the derivation of the formula for p (Example 6) and would have been less useful. So, unless there is a need to use the formula for s_2 a multiple number of times, the derivation is not recommended.

Solution to Scenario Question

Solution to Scenario Question

Algebraically, solve for R^3.

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Given equation.

$$\frac{1}{R_T} - \frac{1}{R_1} - \frac{1}{R_2} = \left(\cancel{\frac{1}{R_1}} + \cancel{\frac{1}{R_2}} + \frac{1}{R_3}\right) - \cancel{\frac{1}{R_1}} - \cancel{\frac{1}{R_2}}$$

Isolate term with R^3.

$$\frac{1}{R_3} = \frac{1}{R_T} - \frac{1}{R_1} - \frac{1}{R_2}$$

Simplify (and reverse sides).

$$\frac{1}{R_3} = \frac{1}{R_T} \cdot \left(\frac{R_1R_2}{R_1R_2}\right) - \frac{1}{R_1} \cdot \left(\frac{R_TR_2}{R_TR_2}\right) - \frac{1}{R_2} \cdot \left(\frac{R_TR_1}{R_TR_1}\right)$$

Get common denominator for fractions on the right side.

$$\frac{1}{R_3} = \frac{R_1R_2 - R_TR_2 - R_TR_1}{R_TR_1R_2}$$

Add fractions and simplify.

$$R_3 = \frac{R_TR_1R_2}{R_1R_2 - R_TR_2 - R_TR_1}$$

Invert to solve for R^3.

$$R_3 = \frac{(14\ \Omega)(33\ \Omega)(28\ \Omega)}{(33\ \Omega)(28\ \Omega) - (14\ \Omega)(28\ \Omega) - (14\ \Omega)(33\ \Omega)}$$

$$= 185\ \Omega$$

Substitute values and solve for the value of R^3.

Practice Exercises

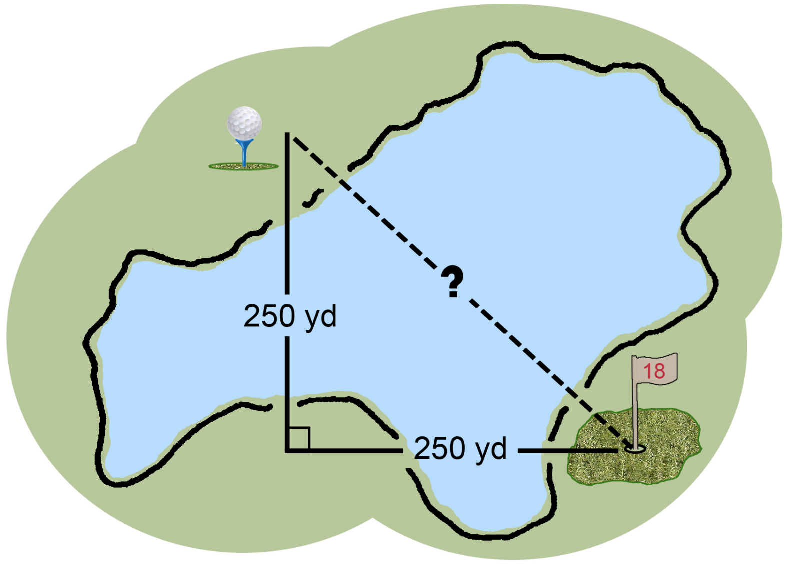
Exercises

Exercise 1

The ratio of the speed of light in a vacuum ($c = 3 \times 10^8$ m/s) to the speed of light in a particular medium (v) results in a constant known as the index of refraction (n). This can also be represented with the formula $n = \frac{c}{v}$. Find the speed of light in glass ($n = 1.5$).

Exercise 2

On a certain golf course, the traditional play is to take two strokes to go around a small lake. Each stroke requires a distance of about 250 yards, as shown below. You wonder how strong a drive would be needed to make it across the lake in one stroke. Determine how far a single stroke would have to go to cross the lake toward the green. (**Hint:** For a right triangle, $a^2 + b^2 = c^2$.)



Exercise 3

Your medical insurance policy requires you to pay the first $100 of your hospital expenses (this is known as a deductible). The insurance company will then pay 80% of the remaining expense. The amount that you must pay can be expressed by the following formula:

$$E = [(T - D) \times (1.00 - P)] + D$$

where E is the expense to you (how much you must pay),
T is the total of the hospitalization bill,
D is the deductible you must first pay,
P is the decimal percentage that the insurance company pays after you meet the deductible.

Suppose you are expecting a short surgical stay in the hospital, for which you estimate the total bill to be about $5000. Use the formula above to estimate the expense to you for the total hospital bill.

Exercise 4

Greenshield's formula can be used to determine the amount of time a traffic light at an intersection should remain green. This formula is shown below.

$$G = 2.1n + 3.7$$

where G is the "green time" in seconds and
n is the average number of vehicles traveling in each lane per light cycle.

Find the green time for a traffic signal on a street that averages 19 vehicles in each lane per cycle.

Exercise 5

White light is incident on the surface of a soap bubble. A portion of the surface reflects green light of wavelength $\lambda_0 = 540$ nm. Assume that the refractive index of the soap film is near that of water, so that $n_f = 1.33$. Estimate the thickness (in nanometers) of the soap bubble surface that appears green in 2nd order (m = 2).

Use $$2n_f t + \frac{\lambda_0}{2} = m\lambda_0$$

Exercise 6

In close-up photography, the distance of the object from the lens determines how far the lens must be from the film. This requires special lenses and focusing mechanisms. These distances (all measured in the same units) are controlled by the formula.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

where f is the focal length of the lens,
p is the distance of the object being viewed from the center of the lens,
and
q is the distance of the image formed on the film from the center of the lens.

a. Rewrite the formula, isolating the variable representing the object distance.

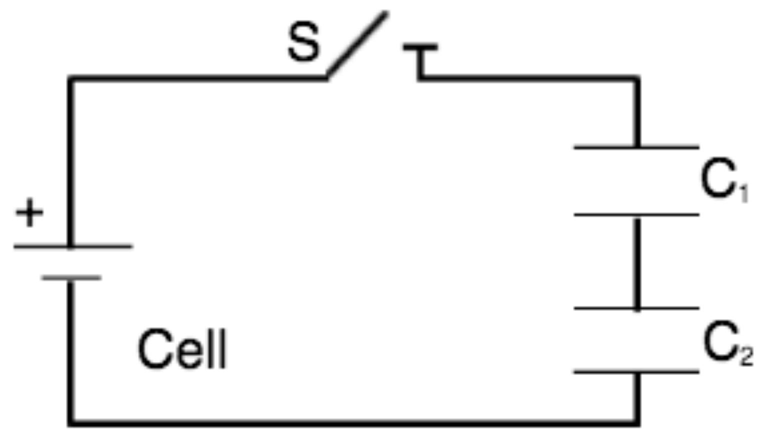
b. Determine the object distance predicted by the equation for a lens with a focal length of 50 mm and a lens to film distance of 6.2 cm.

Exercise 7

Two capacitors are set up in series. Find the equivalent capacitance, C_{eq} , where $C_1 = 2.3\mu F$ and $C_2 = 6.5\mu F$, using the formula below:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

(Remember that μ stands for $\times 10^{-6}$.)

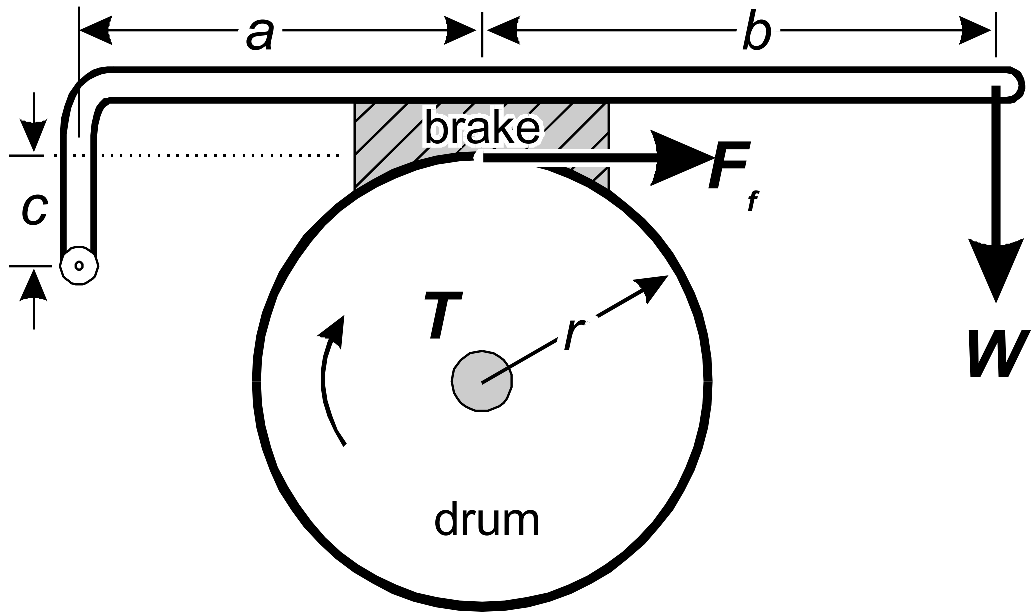


Exercise 8

A block brake mechanism is being fabricated to counteract a 20-inch diameter rotating drum shaft. The brake block is made of oak and the drum surface is cast iron, yielding a coefficient of friction between the two surfaces of $\mu = 0.30$. The block is positioned 24 inches from the end of a 40-inch brake arm (i.e., a = 16, b = 24). A piston attached to the right end of the brake arm can supply a force of 200 pounds. What length of offset bracket (length c) is needed to provide a counteracting torque of 1800 in-lb?

$$T=\frac{Wr(a+b)}{\left(\frac{a}{\mu}-c\right)}$$

where T = torque (inch-pounds)
W = force applied to brake lever by piston(pounds)
r = radius of brake drum (inches)
μ = coefficient of friction between drum and brake
a, b, c = dimensions of brake arm (inches)



Torque ***T*** produced by block brake

Solutions to Practice Exercises

1.

$$n = \frac{c}{v} \triangleleft$$

$$1.5 = \frac{3 \times 10^8 \text{ m/s}}{v}$$

$$v \times 1.5 = \frac{3 \times 10^8 \text{ m/s}}{\cancel{v}} \times \cancel{v}$$

$$1.5v = 3 \times 10^8 \text{ m/s}$$

$$\frac{\cancel{1.5}v}{\cancel{1.5}} = \frac{3 \times 10^8 \text{ m/s}}{1.5}$$

$$v = 2 \times 10^8 \text{ m/s}$$

The speed of light in glass is 2×10^8 m/s.

2.

$$a^2 + b^2 = c^2$$

$$(250 \text{ yd})^2 + (250 \text{ yd})^2 = c^2$$

$$125{,}000 \text{ yd}^2 = c^2$$

$$\sqrt{125{,}000 \text{ yd}^2} = \sqrt{c^2}$$

$$354 \text{ yd} = c \quad \text{(rounded)}$$

c is the hypotenuse of the right triangle.

The hole is about 350 yards away. To clear the water, the ball does not have to go 350 yards in the air, but if it gets over the water it will probably roll that far.

3.

$$E = \left[(T - D) \times (1.00 - P)\right] + D$$

$$E = \left[(\$5000 - \$100) \times (1.00 - 0.8)\right] + \$100$$

$$E = (\$4900 \times 0.2) + \$100$$

$$E = \$980 + \$100$$

$$E = \$1080$$

You will pay $1080 for the hospital bill.

4.

$G = 2.1n + 3.7$
$G = 2.1(19) + 3.7$
$G = 39.9 + 3.7$
$G = 43.6$

The light should stay green for about 44 seconds.

5.

$$2n_f t + \frac{\lambda_0}{2} = m\lambda_0$$

$$2n_f t + \frac{\lambda_0}{2} - \frac{\lambda_0}{2} = m\lambda_0 - \frac{\lambda_0}{2}$$

Factoring out simplifies the result, but is not necessary

$$2n_f t = \lambda_0\left(m - \frac{1}{2}\right)$$

$$\frac{2n_f t}{2n_f} = \lambda_0\left(m - \frac{1}{2}\right) \times \frac{1}{2n_f}$$

$$t = \frac{\lambda_0}{2n_f}\left(m - \frac{1}{2}\right)$$

$$t = \frac{540 \text{ nm}}{2(1.33)}\left(2 - \frac{1}{2}\right)$$

$$t = \frac{540 \text{ nm}}{2.66}(1.5)$$

$$t = 305 \text{ nm}$$

6. a.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p} + \frac{1}{q} - \frac{1}{q}$$

$$\frac{q}{fq} - \frac{f}{fq} = \frac{1}{p}$$

$$\frac{q-f}{fq} = \frac{1}{p}$$

$$p \times \left(\frac{q-f}{fq}\right) = \frac{1}{\cancel{p}} \times \cancel{p}$$

$$p \times \left(\frac{\cancel{q-f}}{\cancel{fq}}\right) \times \left(\frac{\cancel{fq}}{\cancel{q-f}}\right) = 1 \times \left(\frac{fq}{q-f}\right)$$

$$p = \frac{fq}{q-f}$$

b.

$$50\ \cancel{mm} \times \frac{1\ cm}{10\ \cancel{mm}} = 5\ cm$$

$$p = \frac{(5\ cm)(6.2\ cm)}{6.2\ cm - 5\ cm}$$

$$p = \frac{31\ cm^2}{1.2\ cm}$$

$$p = 25.8\ cm$$

7.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{2.3\ \mu F} + \frac{1}{6.5\ \mu F}$$

$$\frac{1}{C_{eq}} = 0.5886 \frac{1}{\mu F}$$

$$C_{eq} = \frac{1}{0.5886 \frac{1}{\mu F}} = 1.7\ \mu F, \text{ or } 1.7 \times 10^{-6} F \text{ (rounded)}$$

Use the 1/x button on the calculator.

8. Given the provided formula for the torque delivered by a block brake, use algebra principles to isolate the variable for the length c.

$$T = \frac{Wr(a+b)}{\left(\frac{a}{\mu} - c\right)}$$ Given equation.

$$T \times \left(\frac{a}{\mu} - c\right) = \frac{Wr(a+b)}{\cancel{\left(\frac{a}{\mu} - c\right)}} \times \cancel{\left(\frac{a}{\mu} - c\right)}$$

Multiply both sides by $\left(\frac{a}{\mu} - c\right)$, to get the term including c out of the denominator.

$$T\left(\frac{a}{\mu}\right) - Tc = Wr(a+b)$$

Use distributive property to multiply terms in parentheses by *T*.

$$\cancel{T}\cancel{\left(\frac{a}{\mu}\right)} - \cancel{T}\cancel{\left(\frac{a}{\mu}\right)} - Tc = Wr(a + b) - T\left(\frac{a}{\mu}\right)$$

Subtract $T\left(\frac{a}{\mu}\right)$ from both sides.

$$\frac{\cancel{-T}c}{\cancel{-T}} = \frac{Wr(a + b) - T\left(\frac{a}{\mu}\right)}{-T}$$

Divide both sides by –T.

$$c = \frac{Wr}{-T}(a + b) + \left(\frac{a}{\mu}\right)$$

$$c = \left(\frac{a}{\mu}\right) - \frac{Wr}{T}(a + b)$$

Simplify and rearrange.

$$c = \left(\frac{16\text{ in}}{0.30}\right) - \frac{(200\text{ lb})(10\text{ in})}{1800\text{ in-lb}}(16\text{ in} + 24\text{ in})$$

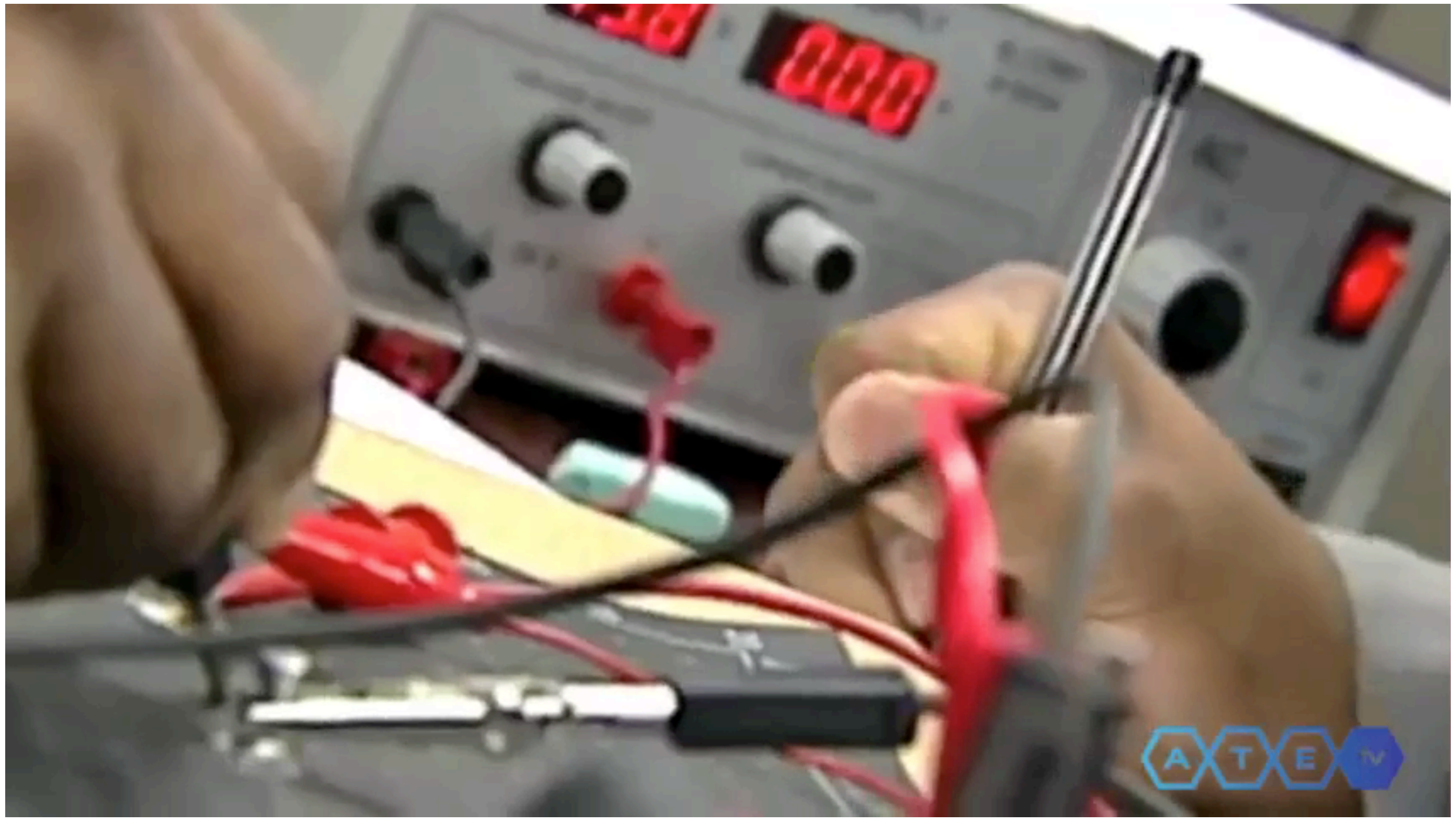
$$c = 8.9\text{ in}$$

Substitute values and solve for c.

So, the offset bracket needs to be 8.9 inches long to deliver the desired braking torque to the drum.

Career Video

Electronic Engineering: Never Too Late Too Learn



An older student learns a new career at Florence-Darlington Technical College; students apply GPS to agriculture at Kirkwood Community College; 3-D printing revolutionizes design at Saddleback College.

Powers and Roots

4

Objectives

When you have completed this section, you should be able to do the following:

1. Simplify expressions with powers and roots
2. Solve for a variable in an equation with powers and roots

Scenario

Resonance in an LC circuit occurs at a frequency f given by

$$f = \frac{1}{2\pi\sqrt{LC}}$$

where f = resonant frequency, in cycles per second (or Hertz)
L = inductance, in Henries
C = capacitance, in farads

What inductance value is required to produce a resonance at 50 cycles per second when placed in series with a 20 μf capacitor?

Before looking at the solution, work through the lesson to further develop your skills in this area.

Powers

A **power** is a short way to write repeated multiplication. The expression 8^4 is a power. The *exponent* 4 represents the number of times the *base* 8 is to be used as a factor.

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{n factors of a}}$$

Example:

$$8^4 = \underbrace{8 \cdot 8 \cdot 8 \cdot 8}_{\text{4 factors of 8}} = 4096$$

The expression " 8^4 " is read "eight to the fourth power." Other powers are displayed in the table below.

Exponential Form	**Words**	**Meaning**	**Value**
5^1	5 to the first power	5	5
6^2	6 to the second power, or 6 *squared*	$6 \cdot 6$	36
4^3	4 to the third power, or 4 *cubed*	$4 \cdot 4 \cdot 4$	64
7^4	7 to the fourth power	$7 \cdot 7 \cdot 7 \cdot 7$	2401

Example 1

The *braking distance* is the distance traveled by a car from the instant the driver applies the brakes until the car comes to a stop. The following formula can be used to estimate braking distance.

$$b = \frac{r^2}{30 \times F}$$

where b is the estimated braking distance in feet,
r is the car's speed in miles per hour, and
F is the driving surface factor, given by the following table.

Driving surface factor		
Type surface	**Dry road**	**Wet road**
Asphalt	0.85	0.65
Concrete	0.90	0.60
Gravel	0.65	0.65
Packed snow	0.45	0.45

a. What braking distance would you estimate for a car traveling on dry asphalt at a speed of 55 miles per hour?

b. At the scene of an accident, a car's skid marks indicate that it required about 215 feet to brake to a complete stop on wet asphalt. Isolate r in the formula and find an estimated speed for the car as it began braking.

Solution

a. Use the formula and substitute r = 55 and F = 0.85.

$$b = \frac{r^2}{30 \times F}$$

$$b = \frac{55^2}{30 \times 0.85}$$

b = 120 ft (rounded)

b. Isolate r in the formula. First multiply both sides of the equation by the denominator.

$$b \times (30 \times F) = \frac{r^2}{\cancel{(30 \times F)}} \bullet \cancel{(30 \times F)}$$

Then, to obtain r, find the square root of both sides.

$$r = \sqrt{b \times 30 \times F}$$

Substitute b = 215 and F = 0.65

$$r = \sqrt{215 \times 30 \times 0.65}$$

r = 65 mph (rounded)

Plus and Minus

When finding the power of a *negative* number, we must be especially careful.

Recall that

$$5^4 = 5 \times 5 \times 5 \times 5 = 625$$

It may seem simple to also note that

$$(-5)^4 = (-5) \times (-5) \times (-5) \times (-5) = 625$$

Now try it in your calculator and see what result you get. If you remembered the parentheses, you did fine; if not, your answer was negative. It is a good idea to always check you answer with your expectations. Your calculator can mislead you. When you type: -5^4 , the calculator essentially reads it as – . Be sure you understand how the calculator is working before you assume that what it is telling you is true. Raising any number (unless it is zero) to an *even* power will always result in a positive answer.

b^p = positive # *where* $b \neq 0$ and p is even.

When you have an *odd* exponent, two things can happen. If the base is positive, the result will remain positive; if the base is negative, the result stays negative.

b^p = positive # *where* $b > 0$ and p is odd

b^p = negative # *where* $b < 0$ and p is odd

When you think about them, these rules are rather self-evident.

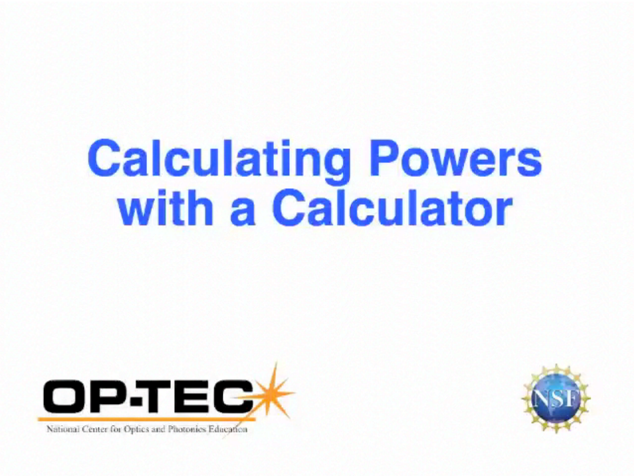
$4^2 = 16$ and $(-4)^2 = 16$

Any number raised to an even power is *positive*.

$4^3 = 64$ Any positive number raised to any power is *positive*.

$(-4)^3 = -64$ Any negative number raised to an odd power is *negative*.

MOVIE 4.1 Calculating Powers with a Calculator



Click or Tap to Watch

Roots

Roots are equal factors of numbers expressed as powers.

$7^2 = 7 \times 7 = 49$

$\sqrt{49} = 7$

The symbol $\sqrt{49}$ asks the question "What number multiplied by itself gives 49?" Or, saying it another way, "What are the two equal factors of 49?"

You can read $\sqrt{49}$ as "the square root of 49." This symbol by itself ($\sqrt{\ }$) always means "the square root."

Your calculator can help you find the roots of numbers. On many calculators, you use the same key to find both the square (x^2) and the square root ($\sqrt{x}$). To use the second meaning of the key, you first press the INVerse key to show that you want the inverse or backward meaning of the key. On some calculators, the INV key may be labeled "2nd F" or just "2nd." This key tells the calculator to use the second meaning of the next key pressed.

Some calculators actually have $\sqrt{x}$ keys.

Example 2

Find the square root of 256.

$\sqrt{256} = ?$

To do this with your calculator,

Enter 256.

(If your calculator requires it) Press the INV (or 2nd) key.

Press the key marked $\sqrt{x}$.

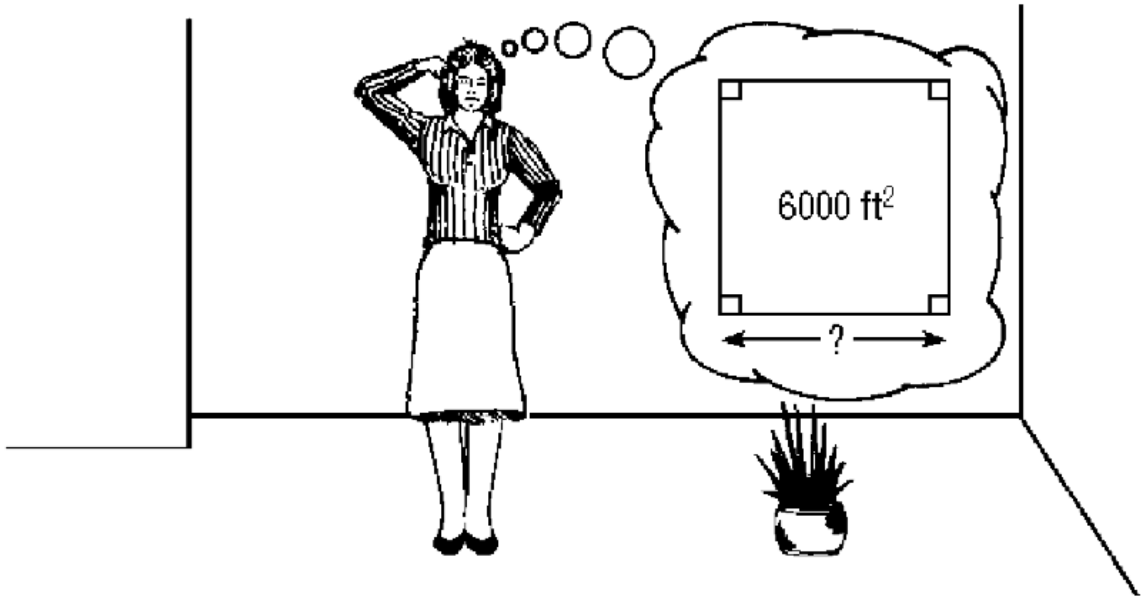
So, $\sqrt{256} = 16$

You can check your answer with the calculator too.

If $\sqrt{256}$ is 16, then 16^2 is 256. Is this true? Use your calculator to find 16^2.

Example 3

The Widget Manufacturing Company needs to build a storeroom that will provide 6000 ft^2 of floor space—as shown below. If the storeroom is going to be square, how many feet on a side will it be?



Square feet of floor space

The formula for the area of a square is

$A = s^2$, where s stands for the side of the square.

Rewrite the formula, substituting what you know for the letters. Since you know the area, the formula now looks like this:

$$6000 \text{ ft}^2 = s^2$$

But you want to find s, not s^2. You should find the square root.

A formula always equates a left side to a right side. To keep the two sides equal, you always do the same thing to both sides.

In this case, you need to find the square root of each side of the formula.

$$\sqrt{6000\ ft^2} = \sqrt{s^2}$$

Since squaring and finding the square root are inverses of each other, $\sqrt{s^2}$ is just plain s.

What is $\sqrt{6000\ ft^2}$ = ? To find out, use your calculator. Write the answer with the correct units. Remember, $\sqrt{ft^2}$ = ft, just as, for example, $\sqrt{4^2} = 4$.

Did you get about 77.46 feet for the side of the square?

Check your answer by squaring it. If you still have the answer in the window of your calculator, just press the x^2 key and you will get back to 6000.

Finding Cube Roots

The root symbol can also be used to mean other roots. You can write a small number near the "bend" of the root symbol to show other roots.

$\sqrt[3]{}$ means "the cube root." (Remember that " x^3 " is known as "x cubed.")

$\sqrt[3]{8}$ asks the question "What are the three equal factors whose product is 8?"

Example 4

What do you think the value of $\sqrt[3]{8}$ is?

$\sqrt[3]{8}$ asks, "If $r \times r \times r = 8$, what number does r stand for?"

Since $2 \times 2 \times 2 = 2^3 = 8$, then $\sqrt[3]{8} = 2$.

Some calculators may have $\sqrt[3]{x}$ keys as well as $\sqrt{x}$ keys.

To find cube roots with a calculator that does not have a $\sqrt[3]{x}$ key, use the y^x key, but this time press the INV key first to get $\sqrt[x]{y}$ or "the xth root of y."

Look at the next example, where a cube root is involved in finding the dimensions of a box.

Example 5

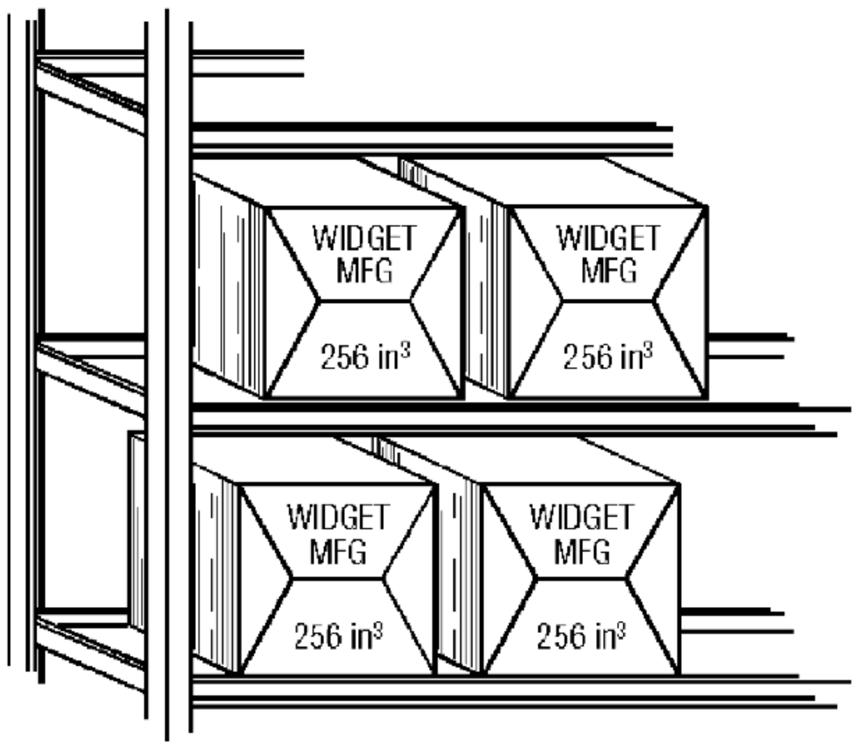
The Acme Storage Company sells a packing box in the shape of a cube that holds 256 cubic inches, as shown below. How long is the edge of the box?

The formula for the volume of a cube is

$V = s^3$, where s stands for any one of the edges of the cube.

Rewrite the formula, substituting what you know for the letters. Since you know the volume, the formula now looks like this:

$$256 \text{ in}^3 = s^3$$



To find s, you find the cube root of both sides of the equation.

Since cubing and finding the cube root are inverses of each other, $\sqrt[3]{s^3}$ is just plain s.

What is $\sqrt[3]{256 \text{ in}^3}$? To find out, use your calculator.

Enter 256.

(Press the INV key or 2nd F key if needed.)

Press the $\sqrt[x]{y}$ key.

Enter 3 (to find the cube root).

Press the = key.

If your calculator has a $\sqrt[3]{x}$ key, you need to do only the following:

Enter 256.

(Press the INV key or 2nd F key if needed).

Press the $\sqrt[3]{x}$ key.

Write the answer with the units. (Remember, $\sqrt[3]{\text{in}^3} = \text{in}$). Did you get about 6.3 inches for the edge of the cube?

You can use the $\sqrt[x]{y}$ to find many different roots by following these steps:

Enter the number.

(Press the INV key or 2nd F key if needed.)

Press the $\sqrt[x]{y}$ key.

Enter the root that you want.

Press the = key.

Solution to Scenario Question

Solution to Scenario Question

We need to solve for the inductance L.

$f = \dfrac{1}{2\pi\sqrt{LC}}$	Given equation.
$\cancel{f} \times \dfrac{\sqrt{LC}}{\cancel{f}} = \dfrac{1}{2\pi\cancel{\sqrt{LC}}} \times \dfrac{\cancel{\sqrt{LC}}}{f}$	Bring desired term involving L out of the denominator, and move f to other side.
$\sqrt{LC} = \dfrac{1}{2\pi f}$	Simplify.
$\left(\sqrt{LC}\right)^2 = \left(\dfrac{1}{2\pi f}\right)^2$	Square both sides to eliminate radical.
$LC = \left(\dfrac{1}{2\pi f}\right)^2$	Simplify.
$\dfrac{L\cancel{C}}{\cancel{C}} = \left(\dfrac{1}{2\pi f}\right)^2 \times \dfrac{1}{C}$	Divide both sides by C to isolate L.
$L = \dfrac{1}{(2\pi f)^2 C}$	Simplify.

So, then substitute given values for f and C, and solve for the value of L.

$$L = \frac{1}{(2\pi f)^2 C}$$

$$= \frac{1}{\left(2\pi(50\ \text{Hz})\right)^2\left(20 \times 10^{-6}\ \text{f}\right)}$$

$= 0.507$ Henries, or 507 mH

Practice Exercises

Exercise 1

A pendulum clock can maintain accurate time because it has a finely adjustable period of swing. The period of swing of a pendulum is given by the formula

$$T = 2\pi r \times \sqrt{\frac{L}{G}}$$

where T is the period (see note below) of the pendulum in seconds,
L is the length of the pendulum in centimeters, and
g is the due to gravity, 980 cm / s^2 .

Note: The period is defined as the time required for one complete swing, back and forth.

a. What would be the period of a pendulum that is 25.00 cm long? (Use your calculator's value for π with this formula.)

b. Suppose you want the same pendulum to have a period of 1.000 second. If the length were shortened to 24.90 cm, would the period be closer to or farther from the desired value of 1.000 second?

Exercise 2

You agree to sell your car for $2600 and allow the buyer to finance it by paying 1% per month interest on the unpaid balance. You would like to amortize the loan in 24 months (finish receiving payment with 24 equal monthly payments). Use the formula below to determine the monthly payment you should ask for.

$$R = P \times \frac{i}{\left[1 - (1 + i)^{-n}\right]}$$

where R is the monthly payment,
P is the loan amount ($2600),
i is the periodic interest rate (0.01 per month), and
n is the number of payments (periods) (24 payments).

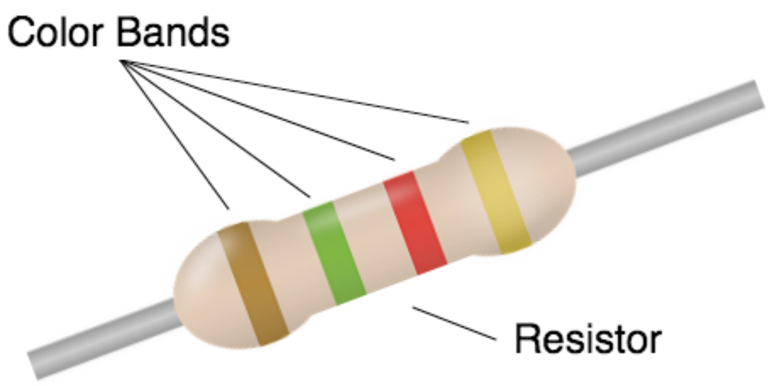
Exercise 3

A shipper uses a cube-shaped box that is advertised to have 700 cubic inches of enclosed volume. What is the largest ball (sphere) one could place in this box?

Exercise 4

Electrical resistors are coded with colored bands to indicate the value of resistance in ohms. Each color represents a number:

Color	Number
Black	0
Brown	1
Red	2
Orange	3
Yellow	4
Green	5
Blue	6
Violet	7
Gray	8
White	9



The resistance value is determined by obtaining the first and second digits from the first and second color bands and multiplying by 10 raised to the *n*th power, where *n* is the number represented by the third color band. For example, a resistor banded red-brown-orange would have a resistance of 21 (from the red and brown bands) 10^3 (from the orange band) ohms, or 21,000 ohms.

Determine the resistance values for the following color band combinations:

a. Green-brown-orange

b. Red-red-blue

c. Black-gray-black

d. Yellow-green-brown

Exercise 5

Some infrared temperature detectors use the fact that energy is radiated from hot "black" objects according to the formula

$$R = \left(5.670 \times 10^{-12}\right) \times \left(273°C + T\right)^4$$

where R is the total energy radiated in watts per cm^2 from a perfect radiator

T is the temperature in degrees Celsius

a. What would be the total energy radiated from such an object that had a temperature of 300°C?

b. What would be the total energy radiated from such an object that had a temperature of 37°C?

Exercise 6

When computing statistics, one must often evaluate the expression "*n!*" or "*n factorial*." (You may have such a key on your calculator.) Factorial expressions are simply a decreasing series of numbers multiplied together. For example, $4! = 4 \times 3 \times 2 \times 1 = 24$. The value of 5! is 120 ($5! = 5 \times 4 \times 3 \times 2 \times 1$). However, manually evaluating factorials of even slightly larger numbers (try 15!) becomes very tedious and results in very large numbers. So, for large values of *n*, Stirling's formula is often used, as shown below.

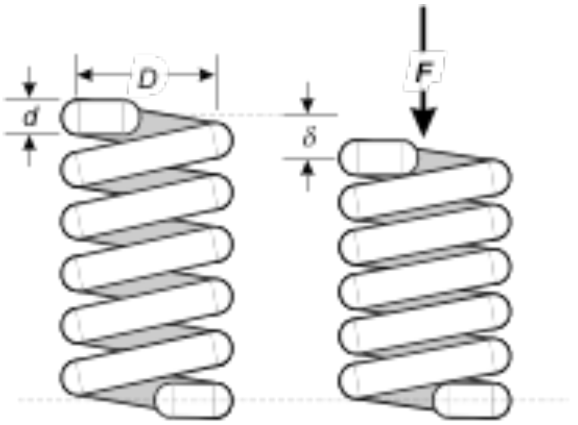
$$n! \approx \sqrt{(2\pi n)} \times \left(\frac{n}{e}\right)^{n}$$

where e is a constant that has a value of about 2.718.

Use Stirling's formula to obtain an approximate value for 30! If your calculator has an x! key, enter 30 and press x! (or 2nd x!) and compare the result with Stirling's formula.

Exercise 7

An automobile engine's valve spring is to be made of #3 gauge high-carbon steel wire (0.2437 in. diameter) having a shear modulus of 11.5×10^6 . Under a load of 100 pounds, the spring should compress no more than ½ inch. If the spring has 5 active turns, what must be the spring diameter?



A force **F** compresses a spring by an amount δ

Equation for spring deflection:

$$\delta = \frac{8FD^3n}{d^4G}$$

where δ = deflection (in.)
F = compression force (lb)
D = spring diameter (in.)
n = number of turns
d = wire diameter (in.)
G = shear modulus

Solutions to Practice Exercises

1. a. $T = 2\pi \times \sqrt{\frac{L}{g}}$

$T = 2\pi \times \sqrt{\frac{25.00 \text{ cm}}{980 \text{ cm/s}^2}}$

T = 1.0035 cm (rounded)

b. $T = 2\pi \times \sqrt{\frac{24.90 \text{ cm}}{980 \text{ cm/s}^2}}$

T = 1.0015 cm (rounded)

The 24.90-cm pendulum is closer to the desired value of 1.000 second.

2.

$$R = P \times \frac{i}{\left[1 - (1 + i)^{-n}\right]}$$

$$R = \$2600 \times \frac{0.01}{\left[1 - (1 + 0.01)^{-24}\right]}$$

$$R = \$2600 \times \frac{0.01}{\left[1 - 0.7876...\right]}$$

$$R = \$122.391 \text{ (rounded)}$$

Therefore, the monthly payments will be $122.39.

Note: The "…" in the problem indicates that those numbers are not being rounded, but kept in the calculator. You should only round at the last step.

3. The volume of a cube is given by the formula $V = s^3$, where *s* is the length of one side of the cube. A sphere of diameter s will just fit within the box. Solve for *s* and substitute the given volume.

$V = s^3$	Given equation.
$\sqrt[3]{V} = \sqrt[3]{s^3}$	Take the cube root of both sides.
$\sqrt[3]{V} = s$	Simplify.
$s = \sqrt[3]{700 \text{ in}^3}$	Substitute value for V and solve.
$= 8.88 \text{ in}$	

So this box could hold a sphere with a diameter as large as 8.88 inches.

4. a. Green-brown-orange

$5\text{-}1\text{-}3 \rightarrow 51 \times 10^3 = 51\,k\Omega$

b. Red-red-blue

$2\text{-}2\text{-}6 \rightarrow 22 \times 10^6 = 22\ M\Omega$

c. Black-gray-black

$0\text{-}8\text{-}0 \rightarrow 08 \times 10^0 = 8\ \Omega$

d. Yellow-green-brown

$4\text{-}5\text{-}1 \rightarrow 45 \times 10^1 = 450\ \Omega$

5. a. $R = (5.670 \times 10^{-12}) \times (273°C + T)^4$

$R = (5.670 \times 10^{-12}) \times (273 + 300)^4$

$R = (5.670 \times 10^{-12}) \times (573)^4$

$R = (5.670 \times 10^{-12}) \times (1.078 \times 10^{11})$

$R = 0.611$ watts/cm^2

b. $R = (5.670 \times 10^{-12}) \times (273 + 37)^4$

$R = (5.670 \times 10^{-12}) \times (310)^4$

$R = (5.670 \times 10^{-12}) \times (9.235 \times 10^9)$

$R = 5.236 \times 10^{-2}$ watts/cm^2 (or 0.05236 watts/cm^2)

6. $n! \approx \sqrt{(2\pi n)} \times \left(\frac{n}{e}\right)^n$

$30! \approx \sqrt{2\pi(30)} \times \left(\frac{30}{2.718}\right)^{30}$

$30! \approx \sqrt{60\pi} \times (11.04...)^{30}$

$30! \approx (13.73...) \times (1.933 \times 10^{31})$

$30! \approx 2.6534 \times 10^{32}$ (rounded)

Entering 30! into a calculator gives a result of 2.6525×10^{32} (rounded). In the “grand scheme of things,” these are pretty close!

7. Using the formula provided for deflection of a helical spring, apply the principles of algebra to isolate the spring diameter variable, D.

$$\delta = \frac{8FD^3n}{d^4G}$$ Given equation.

$$\delta \times \frac{d^4G}{8Fn} = \frac{\cancel{8F}D^3\cancel{n}}{\cancel{d^4G}} \times \frac{\cancel{d^4G}}{\cancel{8Fn}}$$

Multiply both sides by $\frac{\cancel{d^4G}}{\cancel{8Fn}}$ to cancel all terms on the right side but D^3.

$$\frac{\delta d^4G}{8Fn} = D^3$$ Simplify.

$$\sqrt[3]{D^3} = \sqrt[3]{\frac{\delta d^4G}{8Fn}}$$ Swap sides (for readability) and take cube root of both sides.

$$D = \sqrt[3]{\frac{\delta d^4G}{8Fn}}$$ Simplify.

Finally, substitute the known values and solve for the value of the spring diameter, D.

$$D = \sqrt[3]{\frac{\delta d^4G}{8Fn}}$$

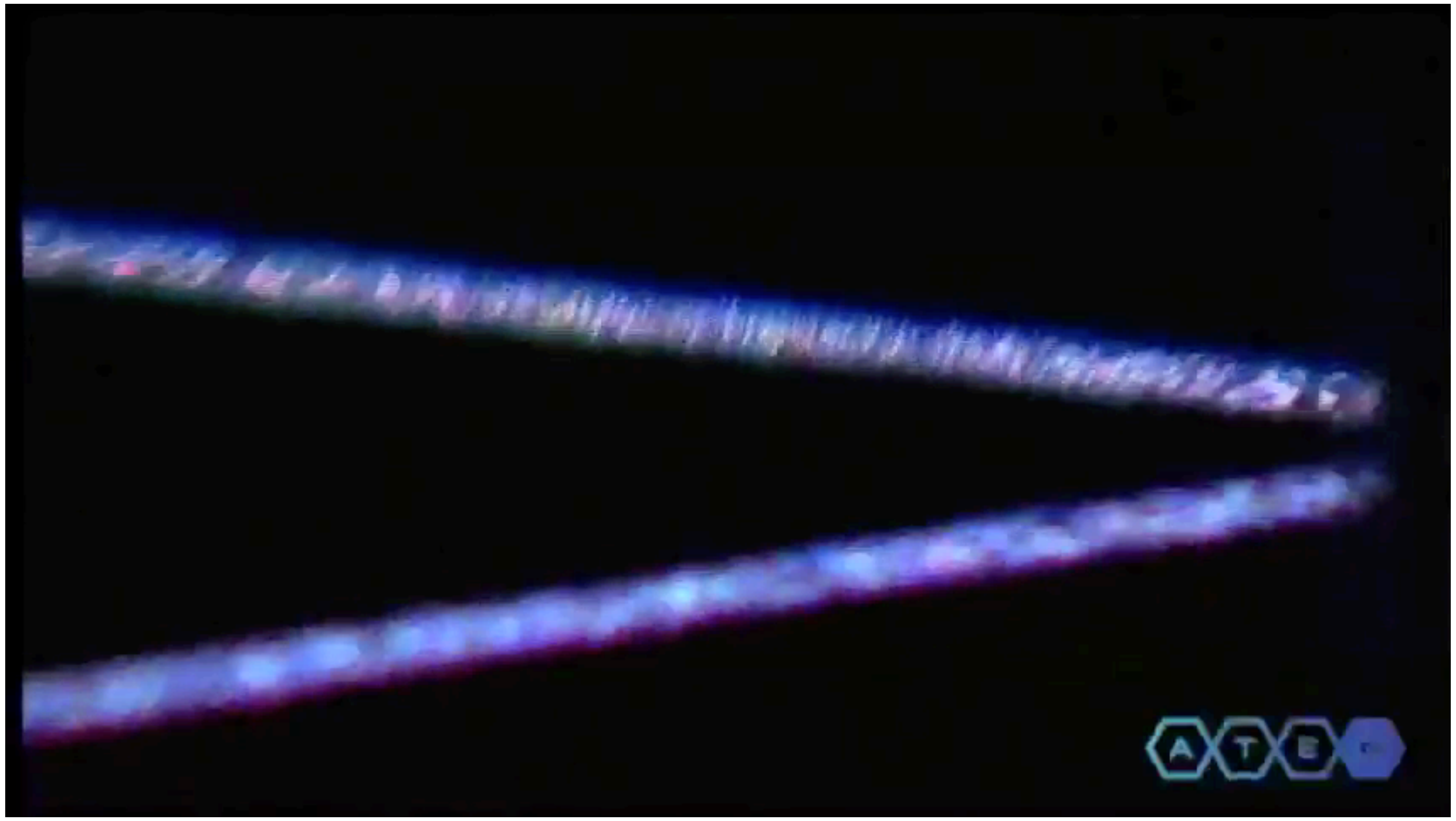
$$D = \sqrt[3]{\frac{(0.500\,in)(0.2437\ in)^4(11.5 \times 10^6\,lb/in^2)}{8(100\,lb)(5)}}$$

$$D = 1.72\ in$$

So the spring should have a diameter of 1.72 inches.

Career Video

The Work Of A Laser Technician



From assembling to troubleshooting, the work of a laser technician varies daily.

Ratio and Proportion

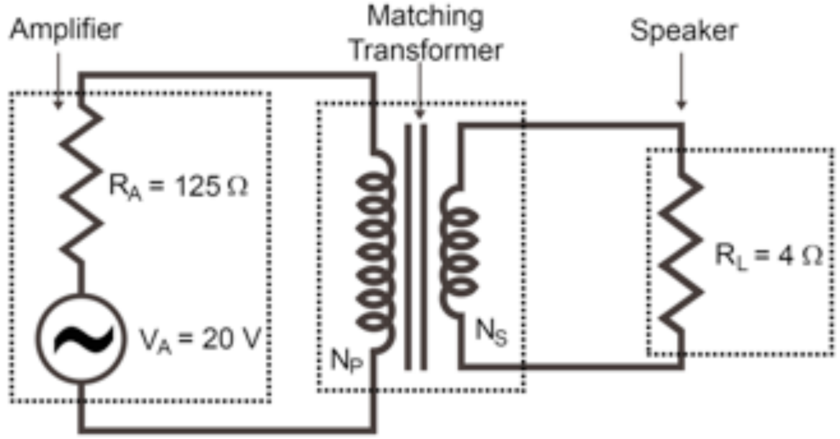
5

Objectives

1. Understand the concepts of ratio, proportion, and percent
2. Solve problems that involve ratio, proportion, and percent

Scenario

An audio amplifier with an open-circuit voltage of 20 V has an internal resistance of 125 Ω. A transformer is needed to match this amplifier to a 4-Ω speaker.



What turns ratio $\frac{N_P}{N_S}$ (primary to secondary) should the matching transformer have? How much more power (what ratio?) can be delivered to the speaker when a matching transformer is used, compared to directly connecting the speaker to the amplifier? (Hint: $\frac{N_P}{N_S} = \frac{V_P}{V_S} = \frac{I_S}{I_P} = \sqrt{\frac{R_P}{R_S}}$, where "P" denotes "primary" and "S" denotes the "secondary.")

Before looking at the solutions, work through the lesson to further develop your skills in this area.

What is a ratio?

What is a Ratio?

A ratio is a quantitative comparison of objects and values. Comparisons of one object to another object, and one value to another value, are made in the following statements. Therefore, these statements describe ratios.

1. Bill is twice as old as Jill. The ratio of Bill's age to Jill's age is 2 to 1. If Bill is 2 years old, Jill is 1 year old. If Bill is 10 years old, Jill is 5 years old. Whatever Jill's age is, Bill's age is twice as much. The ratio 2 to 1 can be written several ways. As a fraction, the ratio is written $\frac{2}{1}$. It also can be written as an indicated division (2 ÷ 1) or with a colon (2 : 1). (Incidentally, if Bill is twice as old as Jill is now, it can never happen again. The difference in their ages remains constant, but the ratio of their ages changes each year.)

2. Donuts are being sold by a supermarket at half price. The ratio of this special selling price to the regular price is 1 to 2. If a box of donuts is now sold for $1, its regular selling price is $2. If an individual donut is now sold for 25 cents, its regular selling price is 50 cents. The ratio of 1 to 2 can be written as the fraction $\frac{1}{2}$. It also can be written as 1 ÷ 2 or as 1 : 2.

3. A large driven gear has 4 times as many teeth as the 10-tooth drive gear. Here, the ratio of "the number of teeth on the driven gear to teeth on the drive gear" is 40 to 10. This ratio can be written as the fraction $\frac{40}{10}$, which can be reduced to $\frac{4}{1}$. The ratio can also be written as 40 ÷ 10, or 4 ÷ 1. With a colon, the ratio can be written as 40 : 10, or 4 : 1.

The chart that follows sums up the ways in which the ratios in these examples can be written. For each of the examples, the objects were given the same units. Bill's and Jill's ages were both given in years. The price of the donuts was given in units of either dollars or cents. The gears were described by the number of teeth on each.

	In words	Fraction	Indicated division	With a colon
Bill is twice as old as Jill.	2 to 1	$\frac{2}{1}$	2 ÷ 1	2 : 1
They're selling donuts at half price.	1 to 2	$\frac{1}{2}$	1 ÷ 2	1 : 2
The driven gear has 4 times as many teeth as the drive gear.	4 to 1	$\frac{4}{1}$	4 ÷ 1	4 : 1

Sometimes the values to be compared don't have the same units. When this happens, the units for one of the values can usually be changed to the units for the other value. Let's look at an example of how to find the ratio of 15 minutes to 1 hour.

First, write the numbers and units as a fraction: $\frac{15 \text{ minutes}}{1 \text{ hour}}$.

Next, either change the units of **minutes** to units of **hours** or change the units of **hours** to units of **minutes**. For this example, we'll change hours to minutes.

$$\frac{15 \text{ min}}{1 \cancel{h} \times \frac{60 \text{ min}}{1 \cancel{h}}} = \frac{15 \cancel{\text{min}}}{60 \cancel{\text{min}}} = \frac{1}{4}$$ (Cancel h and min units.)

Notice that, when the values to be compared both have the same units, the units cancel in the ratio, leaving only a number. Also, the number is here reduced to its lowest terms.

What is a percent?

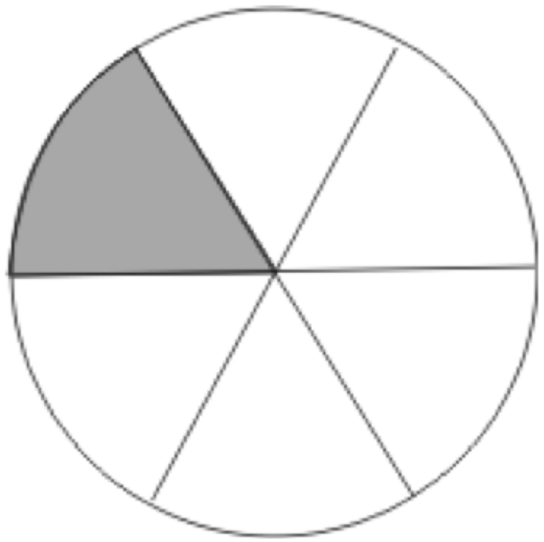
What is a percent?

Now let's look at a type of ratio called percent. Percentage is a comparison of a part of something to the whole of the same thing. The whole is assumed to consist of 100 equal parts.

Percent is just the comparison of a certain number to the number 100. If you have 10 red pencils and 90 yellow pencils, 10 out of the total number of pencils (100) are red. The ratios $\frac{10}{100}$ and 10 : 100 are equal to 10 percent. Since 90 of the pencils are yellow, the ratios $\frac{90}{100}$ and 90 : 100 are equal to 90 percent.

When writing a number as a percent, the symbol "%" is usually used in place of the word percent. This means that a value such as 10 percent usually is written as 10%.

FIGURE 5.1 Circle is divided into 6 equal parts.



Sometimes it's necessary to change a ratio, such as 1 : 6, to a percentage. For example, Figure 5.1 is a circle that's divided into 6 equal parts. One part is shaded. What percentage of the circle is shaded?

Since one part is shaded and there are six parts in the circle, you can write the ratio as $\frac{1}{6}$. To change this ratio to a percentage, divide the number on top (numerator) by the number on the bottom (denominator) to get a decimal number. Then change the decimal number to a "percent" by multiplying by 100%. All this is easy to do with the help of a calculator. Simply divide the number 1 by the number 6. Then multiply the calculator answer (0.167) by 100%.

$0.167 \times 100\% = 16.7\%$

This answer tells us that 16.7% of the circle is shaded.

What is a proportion?

What is a Proportion?

So far, we've shown that a *ratio* is simply a comparison of one value to another value. You also saw about a type of ratio called *percent*. A percentage compares one part of something to 100 equal parts of the same thing. Now let's learn about one more important type of ratio. It's called a *proportion*.

Proportion is the relationship of one ratio, such as $\frac{4}{2}$, to another ratio of equal value, such as $\frac{10}{5}$. The phrase "of equal value" is important. For the ratio $\frac{4}{2}$, 4 divided by 2 equals 2. For the ratio $\frac{10}{5}$, 10 divided by 5 equals 2. These ratios have equal value. Therefore, they are said to be proportional.

The ratio $\frac{4}{2}$ is also proportional to $\frac{8}{4}$, $\frac{12}{6}$, $\frac{20}{10}$ —and so on. The ratio $\frac{2}{4}$ is proportional to ratios such as $\frac{5}{10}$, $\frac{3}{6}$, and $\frac{8}{16}$ because they all have equal values. If the ratios don't have equal values, they're not proportional.

"Constant of proportionality" is a term used in technology. This term is often shortened to just "constant." As you've seen, when a ratio is proportional to another ratio, the ratios have equal values. Since the values don't change, they're said to be "constant."

For example, a spring's strength can be rated by its "spring constant." A spring constant is the ratio of force needed to stretch the spring a certain distance to the *distance* stretched $\left(\frac{f_1}{d_1}\right)$. It's also equal to the ratio of the force needed to stretch the spring another distance to the new distance $\left(\frac{f_2}{d_2}\right)$.

These ratios are proportional. Therefore, they're each equal to a constant. Let's see how that works with an example.

Let's say that, for a certain spring, a force of 10 pounds is needed to stretch the spring 2 inches and a force of 20 pounds stretches the spring 4 inches. The ratios can be written as:

$$\left[\frac{f_1}{d_1} = \frac{f_2}{d_2}\right] \rightarrow \left[\frac{10\,\text{lb}}{2\,\text{in}} = \frac{20\,\text{lb}}{4\,\text{in}}\right] \rightarrow \left[5\frac{\text{lb}}{\text{in}} = 5\frac{\text{lb}}{\text{in}}\right]$$

The constant value of these ratios is 5 lb/in. Therefore, the spring constant is 5 lb/in. The force needed to stretch or compress a spring, or the distance a spring moves when a known amount of force is applied, can be determined if you know the spring constant.

Solution to Scenario Question

Solution to Scenario Question

For a matched condition, $R_P = R_A = 125\ \Omega$ and $R_S = R_L = 4\ \Omega$. So, to find the ratio $\frac{N_P}{N_S}$, substitute the values for the primary and secondary loads.

$$\frac{N_P}{N_S} = \sqrt{\frac{R_P}{R_S}}$$

$$= \sqrt{\frac{125\ \Omega}{4\ \Omega}}$$

$$= 5.59$$

In a matched state, the V_A will be divided equally between the amplifier R_A and the transformer R_P . That is, 10 V by the amplifier and 10 V on the primary. Hence, the current I_P supplied by the amplifier to the primary will be

$$I_P = \frac{V_P}{R_P} = \frac{10\ V}{125\ \Omega} = 0.080\ A$$

And the current supplied by the secondary can be found from the inverse proportion. (Notice that the subscripts are “reversed.” This is why it’s called an *inverse proportion*.)

$$\frac{I_S}{I_P} = \frac{N_P}{N_S}$$

Solve for I_S .

$$I_S = I_P \cdot \frac{N_P}{N_S} = (0.080\ \text{A}) \cdot (5.59) = 0.447\ \text{A}$$

And we can calculate the power delivered by the matched transformer.

$$P_S = (I_S)^2 R_S$$

$$= (0.447\ \text{A})^2 (4\,\Omega)$$

$$= 0.80\ \text{W}$$

If the speaker were connected directly to the amplifier (no matching transformer), we can calculate the current…

$$I = \frac{V}{R} = \frac{20\ \text{V}}{125\ \Omega + 4\ \Omega} = 0.155\ \text{A}$$

and the resulting power…

$$P = (I)^2 R_S$$

$$= (0.155\ \text{A})^2 (4\,\Omega)$$

$$= 0.096\ \text{W}$$

This shows that we get about 8 times ($\frac{0.80\ \text{W}}{0.096\ \text{W}} = 8.33$) more power output when using the matching transformer.

Practice Exercises

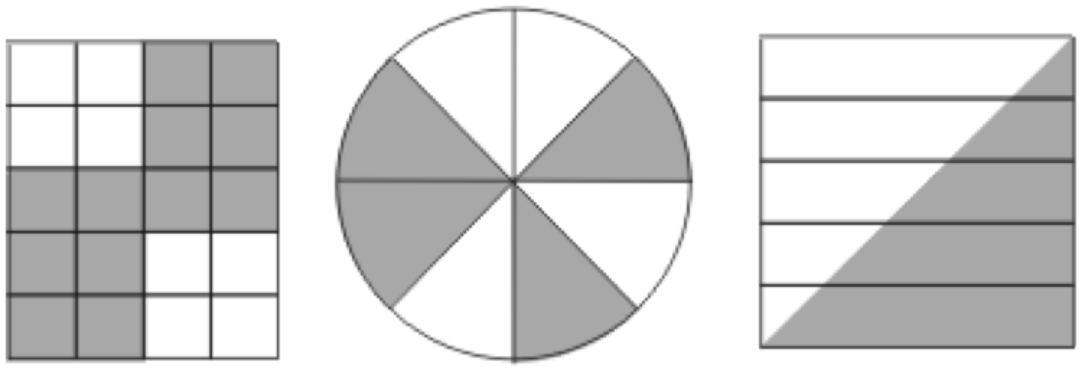
Exercise 1

Read the statements in the following chart. Then write the ratios described by the statements in the blanks. Write each ratio as (1) a fraction, (2) a quotient, and (3) with a colon.

Statement	Fraction	Indicated division	Colon
a. Jana is three times as tall as Mark.			
b. A gallon of milk costs twice as much as a gallon of gasoline.			
c. Her brother is half as old as my brother.			

Exercise 2

What is the ratio of shaded to unshaded areas in the following figures?



a. b. c.

Exercise 3

What percentage of each figure in Exercise 2 is shaded?

Exercise 4

What percentage of each figure in Exercise 2 is unshaded?

Exercise 5

Does adding the percentages in Exercises 3 and 4 give 100% for each figure?

Exercise 6

Solder is a mixture of lead and tin. "Soft" solder has 6 parts tin and 4 parts lead.

a. How many grams of tin are in 1 kilogram of soft solder? (**Hint:** 6 parts + 4 parts = 10 parts = whole.)

b. How many grams of lead are in 1 kilogram of soft solder?

Exercise 7

Which of the following ratios is proportional to $\frac{40}{5}$?

a. $\frac{20}{10}$

b. $\frac{8}{1}$

c. $\frac{5}{40}$

d. $\frac{80}{11}$

Exercise 8

Which of the following ratios is proportional to $\frac{7}{63}$?

a. $\frac{14}{126}$

b. $\frac{1}{8}$

c. $\frac{1}{7}$

d. $\frac{9}{1}$

Exercise 9

What is the constant of the ratios $\frac{12}{1}$ and $\frac{144}{12}$?

Exercise 10

What is the constant of the ratios $\frac{1}{8}$ and $\frac{8}{64}$?

Exercise 11

A force of 70 newtons compresses a spring 2 cm. A second force compresses the same spring only 1 cm. How much force is applied the second time?

(**Hint:** Use the proportion $\frac{70\text{ N}}{2\text{ cm}} = \frac{x(\text{N})}{1\text{ cm}}$.)

Exercise 12

Blueprints and floor plans of buildings are not drawn actual size but are "scaled down." For example, a 1-foot distance in the building may appear as a $\frac{1}{4}$ " distance on the drawing.

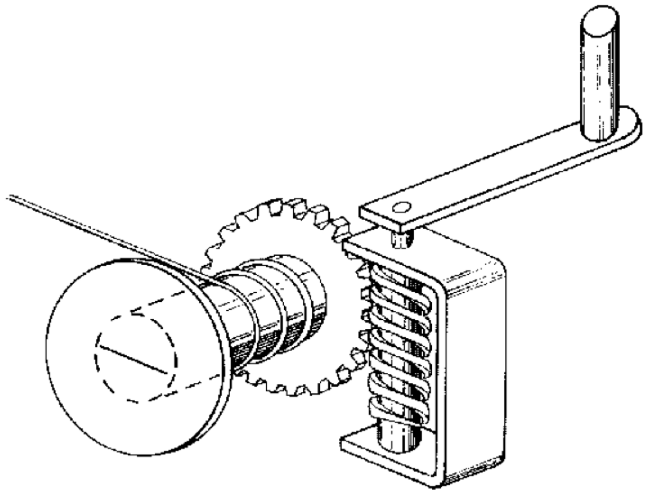
a. What is the ratio that describes the scale used in such a drawing? (Show the ratio both with and without dimensions.)

b. How long a line on a drawing would be used to represent a wall that is 24' long?

c. How long is a duct that is depicted on a drawing by a $9\frac{1}{2}$ -inch line?

Exercise 13

A catalog lists the characteristics of several manually operated winches. One of the features listed is the gear ratio. The worm gear shown below is advertised to have a gear ratio of 41 : 1. That is, 41 turns of the hand crank are needed to produce one turn of the large gear.

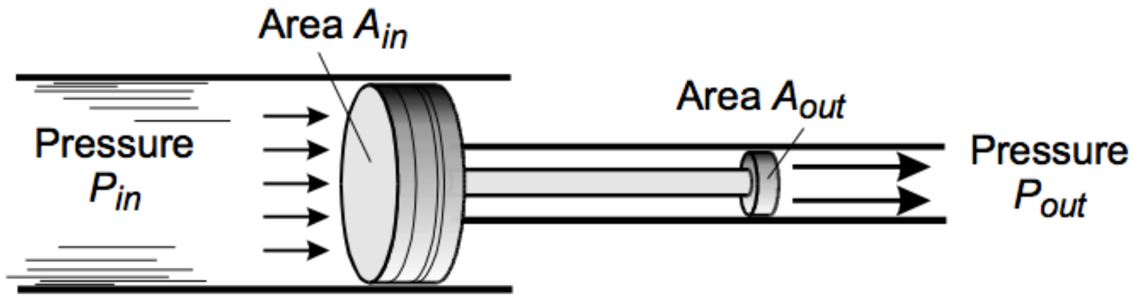


a. The drum attached to the large gear has a diameter of $1\frac{1}{2}$ " and is used to reel in a length of cable. About how many times would the hand crank have to be turned to reel in 18" of cable? (Round to the nearest whole turn of the crank.)

b. Suppose you are able to turn the hand crank at a rate of 40 turns per minute. About how many minutes would it take to wind 18" of cable? (Round to the nearest 0.1 minute.)

Exercise 14

A pressure intensifier is used to convert available air (pneumatic) pressure to high hydraulic pressures, such as needed in hydraulic lifts and actuators. The ratio of input piston areas to output piston areas, is *inversely proportional* to the ratio of input to output pressures.



A water-jet cutting mechanism requires 10,000 psi water pressure at the output. The manufacturing plant can supply 80 psi air pressure as input to a pressure intensifier. If the circular input piston can be no larger than 12 in. diameter, what must be the diameter of the circular output piston to achieve the desired output pressure?

Solutions to Practice Exercises

1.

Statement	Fraction	Indicated division	Colon
a. Jana is three times as tall as Mark.	$\frac{3}{1}$	3 ÷ 1	3 : 1
b. A gallon of milk costs twice as much as a gallon of gasoline.	$\frac{2}{1}$	2 ÷ 1	2 : 1
c. Her brother is half as old as my brother.	$\frac{1}{2}$	1 ÷ 2	1 : 2

2. a. $\frac{12}{8} = \frac{3}{2}$

   b. $\frac{4}{4} = \frac{1}{1}$

   c. $\frac{1}{1}$

3. a. $\frac{12}{20} = \frac{3}{5} = 0.6 \Rightarrow 60\%$

   b. $\frac{4}{8} = \frac{1}{2} = 0.5 \Rightarrow 50\%$

   c. $\frac{1}{2} = 0.5 \Rightarrow 50\%$

4. a. $\frac{8}{20}=\frac{2}{5}=0.4 \Rightarrow 40\%$

b. $\frac{4}{8}=\frac{1}{2}=0.5 \Rightarrow 50\%$

c. $\frac{1}{2}=0.5 \Rightarrow 50\%$

5. Yes

6. a. $\frac{\text{6 parts tin}}{\text{10 parts total}} \times 1\text{ kg} = 0.6\text{ kg}$

$0.6\ \cancel{\text{kg}} \times \frac{1000\text{ g}}{1\ \cancel{\text{kg}}} = 600\text{ g}$ of tin in 1 kg of soft solder

b. $\frac{\text{4 lead}}{\text{10 total}} \times 1\text{ kg} = 0.4\text{ kg}$

$0.4\ \cancel{\text{kg}} \times \frac{1000\text{ g}}{1\ \cancel{\text{kg}}} = 400\text{ g}$ of lead in 1 kg of soft solder

7. b.

8. a.

9. 12

10. $\frac{1}{8}$

11. $\frac{70\text{ N}}{2\text{ cm}} = \frac{x}{1\text{ cm}}$

$70\text{ N}\cdot\text{cm} = (2\text{ cm})x$ Use cross multiplication

$x = 35\text{ N}$

12. a. $1' : \frac{1}{4}''$

$12'' : \frac{1}{4}''$ Restate as a ratio with the same units.

48 : 1 Multiply both sides by 4 to get whole numbers.

b. $\frac{48}{1} = \frac{24'}{x}$ Using 48 : 1 ratio.

$48x = 24' \cdot 1$
$x = 0.5'$, or $6''$

A 6" line on the paper would represent a 24' wall.

c. $\frac{48}{1} = \frac{x}{9.5''}$

$1 \cdot x = 48 \cdot 9.5''$
$x = 456''$, or $38'$

So, a $9\frac{1}{2}''$ line on the paper corresponds to 38' in full scale.

13. a. Diameter = 1.5" Circumference = 1.5(π)"

Each turn of the large gear reels in 1.5(π)" of cable. Since 18" is needed, $\frac{18"}{1.5"\pi}$ = 3.82 turns (rounded) of the *large gear* are required.

$$\frac{\text{41 turns of the small gear}}{\text{1 turn of the large gear}} = \frac{x}{\text{3.82 turns of the large gear}}$$

Solving for x, we find x = 157 turns of the small gear (rounded).

b. $$\frac{\text{40 turns}}{\text{1 minute}} = \frac{\text{157 turns}}{x}$$

$$(\text{40 turns})x = (\text{157 turns})(\text{1 minute})$$

x = 3.9 minutes (rounded)

It will take about 3.9 minutes.

14. The circular input piston will have an area of $A_{in} = \pi(\text{radius})^2 = \pi\left(\frac{12\text{ in}}{2}\right)^2 = 113\text{ in}^2$. The ratio of pressures is inversely proportional to the ratio of the areas. Thus,

$$\frac{\text{Input pressure}}{\text{Output pressure}} = \frac{\text{Output area}}{\text{Input area}}$$

$$\frac{P_{in}}{P_{out}} = \frac{A_{out}}{A_{in}}$$

$$\frac{\text{80 psi}}{\text{10,000 psi}} = \frac{A_{out}}{113\text{ in}^2}$$

$$A_{out}(\text{10,000 psi}) = (\text{80 psi})\left(113\text{ in}^2\right)$$

$$A_{out} = \frac{(80\text{ }\cancel{\text{psi}})\left(113\text{ in}^2\right)}{10{,}000\text{ }\cancel{\text{psi}}}$$

$$A_{out} = 0.904\text{ in}^2$$

The area of the output piston should be a little less than 1 square inch. Next, calculate the diameter of the output piston using the formula for the area of a circle.

$$A_{out} = \pi(radius_{out})^2$$

$$A_{out} = \pi\left(\frac{d_{out}}{2}\right)^2$$

$$\frac{A_{out}}{\pi} = \left(\frac{d_{out}}{2}\right)^2$$

$$\sqrt{\frac{A_{out}}{\pi}} = \frac{d_{out}}{2}$$

$$d_{out} = 2 \cdot \sqrt{\frac{A_{out}}{\pi}}$$

$$d_{out} = 2 \cdot \sqrt{\frac{0.904in^2}{\pi}} = 1.07in$$

So, the output piston must have a diameter of a bit more than an inch.

Career Video

Getting Up To Speed With Wind Energy

Students prepare for a career in wind energy with hands-on experience and daily work using real equipment from the field.

Exponents and Logarithms

6

Objectives

When you have completed this section, you should be able to do the following:

1. Simplify problems involving natural and common logarithms
2. Convert exponential equations to logarithmic equations

Scenario

Several automotive devices rely on resistor-capacitor circuits to achieve adjustable time delays. For example, the intermittent windshield wipers include a variable resistor to adjust the amount of delay between "wipes." As the capacitor charges, the voltage across the capacitor slowly increases according to the formula:

$$E = (12\ V)\left(1 - e^{-\frac{t}{RC}}\right)$$

where E = voltage, in volts, across the capacitor at a time t, in seconds

R = resistor value, in ohms

C = capacitor value, in farads

A certain intermittent wiper circuit is designed to "trigger a wipe" when the voltage E reaches 10.0 V, at which point the capacitor is instantly discharged and the charging process starts all over again. This resistor-capacitor circuit has a capacitor with C = 100 μf.

What resistance value R would produce a trigger every 1.0 second? every 2.5 seconds? every 5.0 seconds?

Before looking at the solutions, work through the lesson to further develop your skills in this area.

The Basics of Exponents and Logarithms

The Basics of Exponents and Logarithms

Thus far we have obtained most of the skills necessary to manipulate equations to solve for variables. However, we have yet to consider the following scenario, which requires solving for *x*.

$$2^x = 3$$

We cannot just move 2 to the other side because x is an exponent here. It is necessary to “bring down” the x. This becomes possible through what is called a **logarithm**. This allows us to rewrite this equation with the x isolated.

$$2^x = 3$$

$\log_2(2^x) = \log_2 3$ Take the “ $\log_2$ ” of both sides of the equation.

$x = \log_2 3$ This “brings down” the exponent of 2.

The above statements are equivalent. In general, we can now write:

$b^x = y$ ⟵ Exponential form

$\log_b(b^x) = \log_b y$

$x = \log_b y$ ⟵ Logarithmic form

where b is the base, x is the exponent, and both y and b are greater than zero. Look carefully at where each of the three letters (x, y, b) goes. Notice that this is a method that solves for the exponent, x, so it should be clear that x ends up being the isolated variable. It is also worth noting that the letter b is the base in both the exponential form and the logarithmic form.

Common and Natural Logarithms

Common and Natural Logarithms

Most of the logarithms you will encounter in photonics will have a base equal to either 10 (common log) or the constant e (natural log). The common log is typically written "log x" and is read "the log of x." When no base is indicated, it is assumed to be 10. The natural log is written "ln x" and is read "the natural log of x." The approximate value of e is 2.7183, although your calculator has the better approximation already stored in memory. The use of these logarithms has been helpful in developing relationships (equations) among many measurable engineering and science characteristics (variables). For example, the following equation relates the variables for a laser in which the amplifier length has a value of L and the mirrors have identical reflectivities R and gain coefficient g.

$$\left(R \cdot e^{Lg}\right)^2 = 1$$

This is a rather complex equation, especially when you need to solve for g. With a little bit of algebra and use of ln properties, it can be reduced to:

$$g = \frac{1}{2L} \ln \frac{1}{R^2}$$

Formulas involving logs are actually rather common. Here's an example of how you might use this one.

Example 1

Consider a HeNe laser in which each mirror reflectivity is 99% (R = 0.99) and the amplifier length is 20 cm (L = 20 cm). Find the gain coefficient, g.

Solution

$$g = \frac{1}{2L}\ln\frac{1}{R^2}$$

$$g = \frac{1}{2(20\text{ cm})}\ln\left(\frac{1}{0.99^2}\right)$$

$$g = \left(\frac{1}{40\text{ cm}}\right)\ln(1.0203...)$$

$$g = 5.03 \times 10^{-4}\text{ cm}^{-1}$$

Example 2

To ensure that you are able to properly use your calculator for log problems, the following short examples are provided. Check that you can get the same answers on your own.

a. log 143 = 2.16 (rounded)

b. ln 65 = 4.17 (rounded)

c. 5 log 6 = 3.89 (rounded)

d. $\frac{2}{3}$ ln 8 = 1.39 (rounded)

e. $\frac{\ln 3}{\ln 2}$ = 1.58 (rounded)

f. log 10,000 = 4

g. log 10 = 1

h. log 1 = 0

i. $\ln e^2 = 2$

j. $\ln e^3 = 3$

Using log and ln in Algebra

As mentioned earlier $y = b^x$ can be written as $\log_b y = x$. More specifically, write y = 10^x as $\log_{10} y = x$ or even simpler as $x = \log y$. Also y = e^x can be represented with $\log_e y = x$ or better with $x = \ln y$.

MOVIE 6.1 Solving Expressions with Exponents

Click or Tap to Watch

Example 3

Solve for x (your answer will include y).

a. $y = 10^{2x}$

b. $y = e^{-3x}$

Solution

a. $2x = \log_{10} y$

$2x = \log y$

$x = \dfrac{\log y}{2}$

b. $-3x = \log_e y$

$-3x = \ln y$

$x = -\dfrac{\ln y}{3}$

Solution to Scenario Questions

Solution to Scenario Questions

Solve the charging equation for the resistance R.

$$E = (12\text{ V})\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\frac{E}{(12\text{ V})} = \frac{\cancel{(12\text{ V})}\left(1 - e^{-\frac{t}{RC}}\right)}{\cancel{(12\text{ V})}}$$

$$\frac{V}{(12\text{ V})} = 1 - e^{-\frac{t}{RC}}$$

$$\cancel{\frac{E}{(12\text{ V})}} - \cancel{\frac{E}{(12\text{ V})}} + e^{-\frac{t}{RC}} = 1 - \cancel{e^{-\frac{t}{RC}}} + \cancel{e^{-\frac{t}{RC}}} - \frac{E}{(12\text{ V})}$$

$$e^{-\frac{t}{RC}} = 1 - \frac{E}{(12\text{ V})}$$

$$\cancel{\ln}\left(\cancel{e}^{-\frac{t}{RC}}\right) = \ln\left(1 - \frac{E}{(12\text{ V})}\right)$$

$$-\frac{t}{RC} = \ln\left(1 - \frac{E}{(12\text{ V})}\right)$$

$$\frac{-\frac{t}{\cancel{R}C} \bullet \cancel{R}}{\ln\left(1 - \frac{E}{(12\text{ V})}\right)} = \cancel{\frac{\ln\left(1 - \frac{E}{(12\text{ V})}\right)}{\ln\left(1 - \frac{E}{(12\text{ V})}\right)}} \bullet R$$

$$R = \frac{-t}{C \bullet \ln\left(1 - \frac{E}{(12\text{ V})}\right)}$$

Then, substitute the values E = 10 V, C = 100 f = 100 10–6 farads, and the three time values to find the resulting R values.

For t = 1.0 sec…

$$R = \frac{-(1.0\,\text{sec})}{\left(100 \times 10^{-6}\text{f}\right) \cdot \ln\left(1 - \frac{10\,\text{V}}{12\,\text{V}}\right)}$$

$$= \frac{-(1.0\,\text{sec})}{\left(100 \times 10^{-6}\text{f}\right) \cdot \ln(-1.792)}$$

$$= 5581\ \text{ohms or } 5.6\,\text{k}\Omega$$

For t = 2.5 sec…

$$R = \frac{-2.5\,\text{sec}}{\left(100 \times 10^{-6}\text{f}\right) \cdot (-1.792)}$$

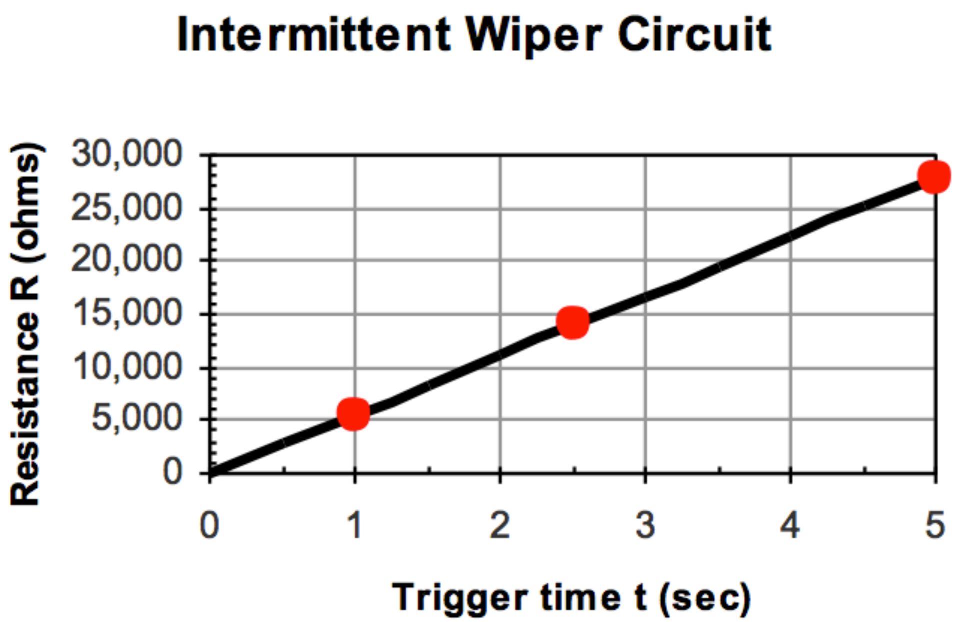
$$= 13{,}952\ \text{ohms, or } 14\ \text{k}\Omega$$

For t = 5.0 sec…

$$R = \frac{-(5.0\,\text{sec})}{\left(100 \times 10^{-6}\text{f}\right) \cdot (-1.792)}$$

$$= 27{,}905\ \text{ohms, or } 28\,\text{k}\Omega$$

…and you can see that for a given capacitor and trigger voltage, the required resistance is a linear function of the desired time to reach 10 volts.



Practice Exercises

Exercise 1

Evaluate the following: $\log 10^0$, $\log 10^1$, $\log 10^2$, $\log 10^3$, $\log 10^4$, and $\log 10^5$. Can you make a generalization about $\log 10^n$?

Exercise 2

Evaluate the following: $\ln e^0$, $\ln e^1$, $\ln e^2$, $\ln e^3$, $\ln e^4$, and $\ln e^5$. Can you make a generalization about $\ln e^n$?

Exercise 3

The optical density (OD) of a material is the degree of opaqueness it has for a given wavelength of light. The higher the OD, the greater the absorption of light. OD is used in reference to light filters, laser goggles, and photographic images. For this problem, we will use an equation that defines OD in terms of transmittance, T. Both OD and T are unitless variables.

$$OD = -\log T$$

A certain red glass absorbs 99%—and transmits 1% (T = 0.01)—of the light entering it when the wavelength of the light is between 400 and 600 nanometers. Find the optical density of the red glass in the 400-to-600-nm range.

Exercise 4

Find the absorption coefficient, a, of a laser propagating at a transmittance of 0.53 through 4.5 cm of the absorbing medium.

$$a = -\frac{1}{x}\ln T$$

where	a	=	Absorption coefficient
	x	=	Distance transmittance is measured into the absorbing medium
	T	=	Transmittance at a distance x into absorbing medium

Exercise 5

Solve for x.

$$y = e^{10x}$$

Exercise 6

Transporting fluids and gases over large distances via pipes involves losses in head pressure due to friction. The calculation of these losses includes a term known as the friction factor, f, which can be approximated by the following equation for values of Reynold's number $R > 4000$:

$$f = \frac{1.325}{\left(\ln\left(\frac{\frac{e}{D}}{3.7} + \frac{5.74}{R^{0.9}}\right)\right)^2}$$

where	f	=	the friction factor for fluid flow in a pipe
	$\frac{e}{D}$	=	is the relative pipe roughness (ratio of bump size to pipe diameter)
	R	=	the Reynolds number for the fluid (related to the fluid viscosity and flow rate).

For R values less than 4000, the flow is treated as "laminar" and the friction factor is considerably simpler:

$$f = \frac{64}{R}$$

a. What is the friction factor f for $\frac{e}{D} = 0.003$ for a fluid with $R = 30{,}000$?

b. Draw a graph of the log f versus log R, typically known as a Moody Chart, for values of $\frac{e}{D} = 0.01,\ 0.001, 0.0001$, and 0.

Solutions to Practice Exercises

1. $\log 10^0 = 0$

   $\log 10^1 = 1$

   $\log 10^2 = 2$

   $\log 10^3 = 3$

   $\log 10^4 = 4$

   $\log 10^6 = 5$

   $\log 10^n = n$

2. $\ln e^0 = 0$

   $\ln e^1 = 1$

   $\ln e^2 = 2$

   $\ln e^3 = 3$

   $\ln e^4 = 4$

   $\ln e^5 = 5$

   $\ln e^n = n$

3. $OD = -\log T$

$OD = -\log(0.01)$

$OD = -(-2)$

$OD = 2$

4. $a = -\frac{1}{x}\ln T$

$a = -\frac{1}{(4.5\ \text{cm})}\ln(0.53)$

$a = 0.141\ \text{cm}^{-1}$

5. $y = e^{10x}$

$10x = \log_e y$

$10x = \ln y$

$x = \frac{\ln y}{10}$

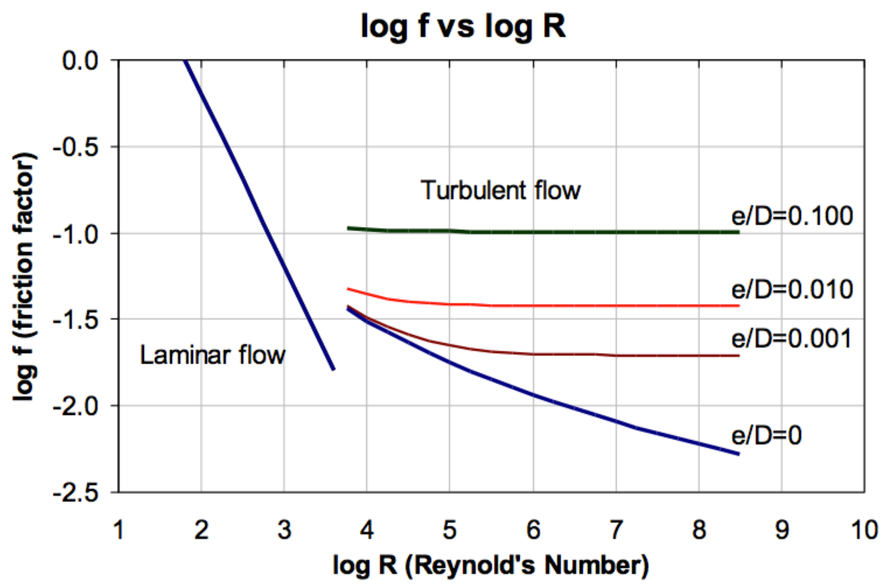
6. a. For $\frac{e}{D} = 0.003$ for a fluid with R = 30,000, we use the more complicated equation for f, substituting in the given values.

$$f = \frac{1.325}{\left(\ln\left(\frac{0.003}{3.7} + \frac{5.74}{(30000)^{0.9}}\right)\right)^2}$$

$$f = \frac{1.325}{\left(\ln(0.001347)\right)^2}$$

$$f = 0.0303$$

b. The graph of the relationship between f and R would appear generally as shown below. Notice that there is a discontinuity in the graph in the region of transition between laminar and turbulent flow. This is typical of the situation when dealing with fluids.



Career Video

Using Your Math And Science

In this video, we learn about environmental technology, internships in biotechnology, and the value of a solid math and science background.

Graphing in Rectangular Coordinates

7

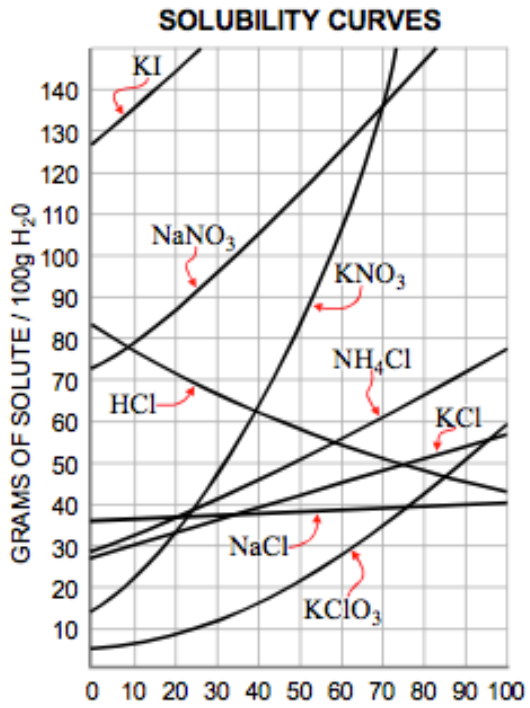
Objectives

When you have completed this section, you should be able to do the following:

1. Identify ordered pairs in the Cartesian coordinate system
2. Read and interpret line graphs in real-world scenarios
3. Draw graphs from a given table of data

Scenario

A chemical engineering technician often refers to relationships depicted by graphs rather than a mathematical formula. For example, you may need to prepare various saturated salt solutions. You do this by dissolving the maximum amount of a given salt that the water can hold. The graphs shown here show the solubility of various salts in water at various temperatures. The solubility on the y-axis gives the grams of salt (solute) per 100 grams of water. The x-axis gives the temperature of the salt-water mixture. Based on the graph:



a. Is it possible to dissolve 50 g of table salt (NaCl) in 100 g of water? If so, at what temperature?

b. Is it possible to dissolve more than 100 g of Potassium Nitrate (KNO_3) in 100 g of water? If so, at what temperature?

c. Approximately how much ammonium chloride (NH_4Cl) will it take to make a saturated solution in 70°C water?

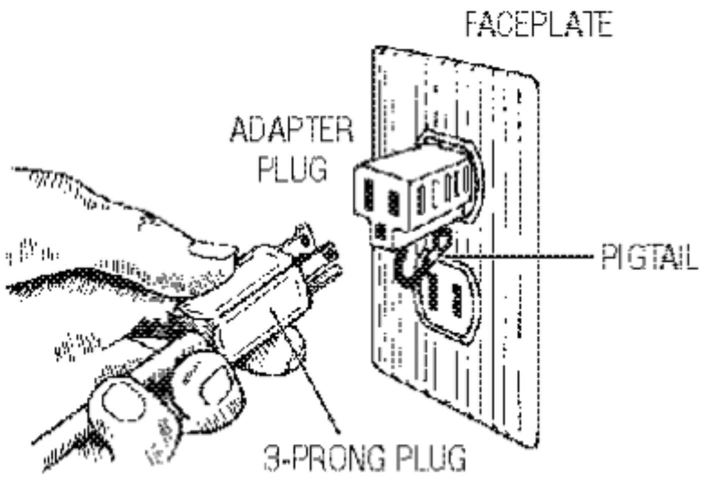
Before looking at the solutions, work through the lesson to further develop your skills in this area.

The Basics of Graphing in Rectangular Coordinates

The figure below explains how to adapt a double-slotted wall outlet to accept a 3-prong plug. The explanation is given in words (part A) and a drawing (part B).

With a double-slotted wall outlet, use an adapter plug. Connect the green tab or pigtail on the adapter plug by first removing the screw in the center of the receptacle faceplate and then putting the screw through the tab and back into the receptacle. Insert the 3-prong plug into the adapter.

A – Words



B – Picture

Words and a picture

Many people find that part B (the picture) gives them information more easily and faster than part A (the words). Most of us grasp information quickly and easily with a picture. Graphs, charts, and tables present numbers and other kinds of information in “picture” form. They provide us with a quicker way to understand the numbers and how they are related.

Cartestion Coordinates

Cartesian Coordinates

The way points and lines are graphed today was first thought of by a French soldier named Descartes over 350 years ago. His idea for picturing points and lines worked so well that the system is named after him—the Cartesian coordinate system.

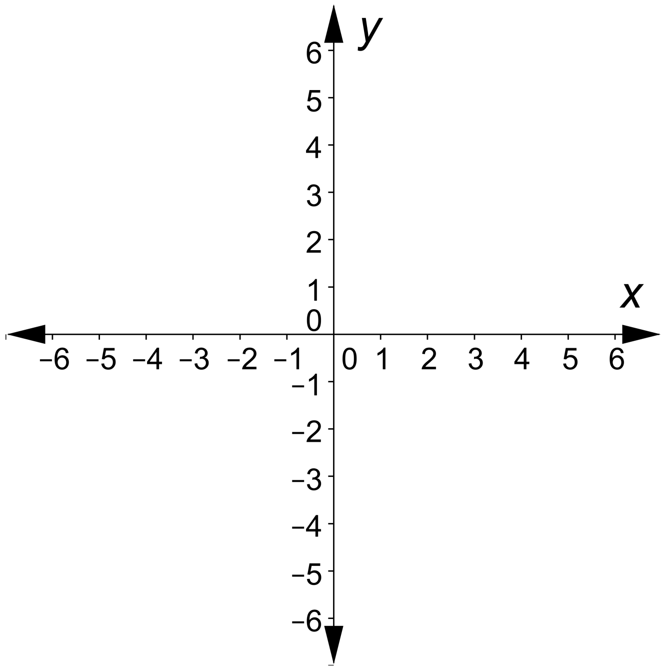
Movie 7.1 The Cartesian Coordinate System

Click or Tap to Watch

Example 1

To draw a Cartesian coordinate system, begin by drawing a number line horizontally in the middle of your paper (if you want to use the whole sheet for the graph). This is the x-axis. Place the zero point (or origin) about halfway across the page. Label this point with the numeral 0.

Then draw another number line perpendicular to the first (making a right angle), crossing the first line at the origin (or zero point). This is the y-axis. Notice that the positive numbers are to the *right* of the origin for the horizontal line and *up* from the origin for the vertical line.



The size of the unit (the distance from zero to one) does not *have* to be the same on both number lines. But it is generally a good idea to draw the units the same unless you have a special reason for making them different. For example, you might need to have many small units on one line and only a few large units on the other.

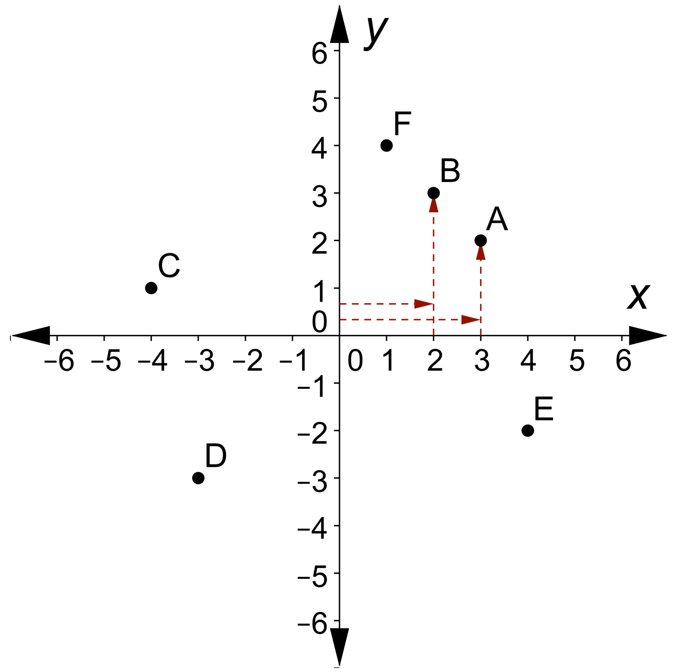
However, once you draw a unit from zero to one on each of the number lines, the size of the unit is always the same on the positive and negative halves of the number lines.

The horizontal number line and the vertical number line are both called axes. The horizontal axis is perpendicular to the vertical axis and they cross at the origin, or zero point.

Every point on the Cartesian coordinate system can be named by a pair of numbers, or coordinates. The first coordinate always tells how far to the right or left of the origin the point is. The second coordinate in the pair always tells how far up or down from the origin the point is. The number pair that names the origin is (0, 0). This is why it is often called the zero point.

Example 2

Write the number pairs that identify the points A, B, C, D, E, and F.



To find the number pair that identifies point A, begin at the origin where the axes cross. Move to the right along the horizontal axis until you are under point A. As you move, count how many units to the right you move from the origin. Write down this number.

Then move up along a line parallel to the vertical axis until you reach point A. As you move, count how many units up you move from the x-axis. Write this number to the right of the first number, place a comma between them, and enclose the pair of numbers in parentheses.

Did you get (3, 2) as the pair of numbers that identifies point A?

To find the pair of numbers that identifies point B, start again at the origin and follow the same process. Did you get (2, 3)?

Does it make any difference which number you write first? Yes, the order of the numbers does make a difference. Point A is (3, 2) and point B is (2, 3)—and they indeed are not the same point.

The pair of numbers that identifies a point in the Cartesian coordinate system is called an ordered pair because the order of the numbers makes a difference.

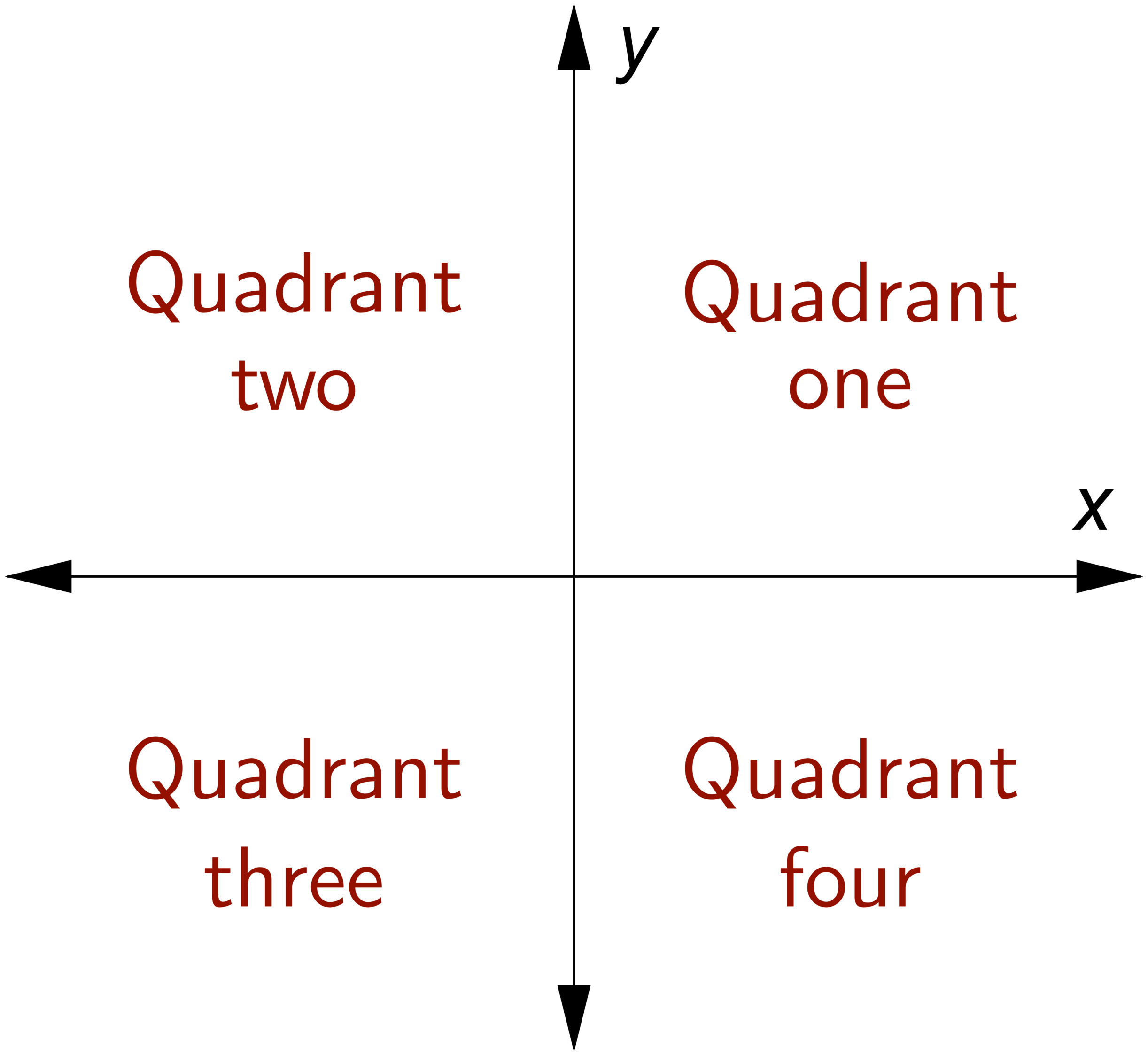
The horizontal axis is called the x-axis, and the vertical axis is called the y-axis. The ordered pair of numbers that identifies any point is (x, y). The x-value of a point is always written first; the y-value is always written second.

The coordinates of the remaining four points are as follows: C (–4, 1), D (–3, –3), E (4, –2), and F (1, 4). Remember: If the point lies to the left of the y-axis, its x-value will be negative. If the point lies below the x-axis, its y-value will be negative.

Quadrants

Quadrants

The Cartesian coordinate system divides flat space like your paper into four quadrants. The figure below shows how these quadrants are numbered.



Example 3

Use the figure above and the ordered pairs that you wrote for Example 2 to help you choose the correct word (positive or negative) to complete each of these statements.

a. All the points in the first quadrant have x-values that are _______ and y-values that are _______.

b. All the points in the second quadrant have x-values that are _______ and y-values that are _______.

c. All the points in the third quadrant have x-values that are _______ and y-values that are _______.

d. All the points in the fourth quadrant have x-values that are _______ and y-values that are _______.

Solution

a. positive, positive

b. negative, positive

c. negative, negative

d. positive, negative

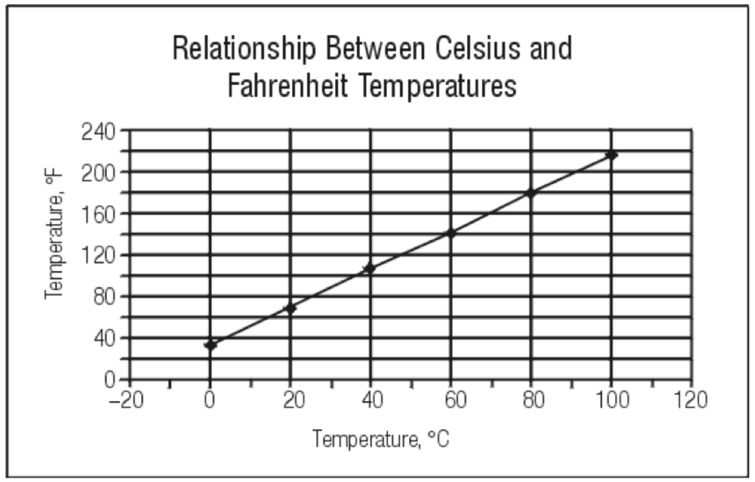
Reading Graphs

Reading Graphs

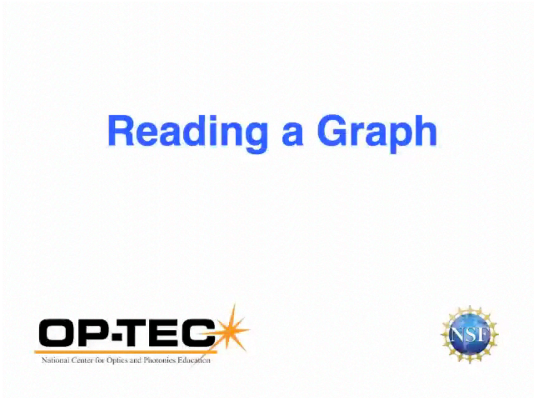
In the Cartesian coordinate system, sometimes points can be connected in a line, sometimes in a curve. Either way, it is important that you understand not just how to plot points, but also how to interpret the graphs they create.

You may have seen thermometers that are labeled with both Celsius and Fahrenheit temperature scales. On these thermometers, you can read the temperature in degrees Fahrenheit or in degrees Celsius. But what if your thermometer has only one temperature scale and you need the other? What if you want to know how the Fahrenheit and Celsius temperature scales are related? You might look for a table that has a matching list of Fahrenheit and Celsius temperatures. Or you might use a conversion formula and a calculator to calculate the value you need. Another way is to use a line graph. The figure below shows a line graph that relates Celsius and Fahrenheit temperatures.

FIGURE 7.1 Relationship between Celsius and Fahrenheit



MOVIE 7.2 Reading a Graph



Click or Tap to Watch

To figure out what the graph contains, first look at the labels. What temperature scale is shown on the bottom axis? What temperature scale is shown on the side axis?

You can see in the graph that the line slopes up, from left to right. This means that for low Celsius temperatures you can expect low Fahrenheit temperatures and for high Celsius temperatures you can expect correspondingly higher Fahrenheit temperatures. The graph is like a table of data. Each dot on the line represents equal Celsius and Fahrenheit temperatures. Let's continue to look at the graph to figure out some equivalent temperatures.

Example 4

What Fahrenheit temperature corresponds to 20°C?

Look at the bottom axis of the graph. Find the place on the axis that represents 20°C. From here, go straight up until you reach the graphed line. There should be a dot there. From that dot go straight left to a location on the side axis. What is the Fahrenheit value represented by this point on the side axis? The point on the side axis may not be a labeled point. In this case, you need to estimate the value, using the adjacent temperature values that are shown. What do you get? Do you get about 70°F? (The exact value is 68°F.)

Example 5

What Celsius temperature corresponds to 32°F?

You can also read a line graph the "other way." Look at the vertical axis of the graph. Find the place on the axis that represents 32°F. From here go straight to the right until you reach the line graph. There should be a dot there. From the dot on the line graph, go straight down until you reach a point on the bottom axis the corresponding Celsius temperature. What is the Celsius value at this point on the axis? Do you get a value of 0°C? Is that what you expected?

What if you want to read a temperature that is not shown with a dot? That's when the line drawn through the dots can help you. The line shows that the dots follow a trend; if more dots were plotted, they would all be on this line. Whenever you use the line to estimate values between the plotted values (dots), you are interpolating between the data. Let's interpolate between the graphed data to find a temperature not shown by a dot.

Example 6

What Fahrenheit temperature corresponds to 50°C?

As before, find the place on the horizontal axis that corresponds to 50°C. Then go straight up until you meet the graphed line. This time you don't find a plotted point — or dot — on the line. However, since the line is drawn, you can assume that a dot is there. From this point, go straight to the left, as before, until you reach the side axis. What Fahrenheit temperature do you find? Do you get a little bit more than 120°F?

Sometimes you need values that are beyond those shown on a graph. You would like to be able to extend the graph, relying on the trend shown by the graph. This is called extrapolating, or extending the graph. Extrapolating normally requires some understanding of how the data on the graph are related. In the case of Fahrenheit and Celsius temperatures, you probably already know that they are related by a simple linear formula. So you know that the relationship or trend shown will continue for higher and higher Celsius and Fahrenheit temperatures (as well as for lower and lower ones). You could sketch in what you expect the graph to look like by extending the line that is already drawn.

Example 7

Find the Fahrenheit temperature that is equal to a temperature of 110°C.

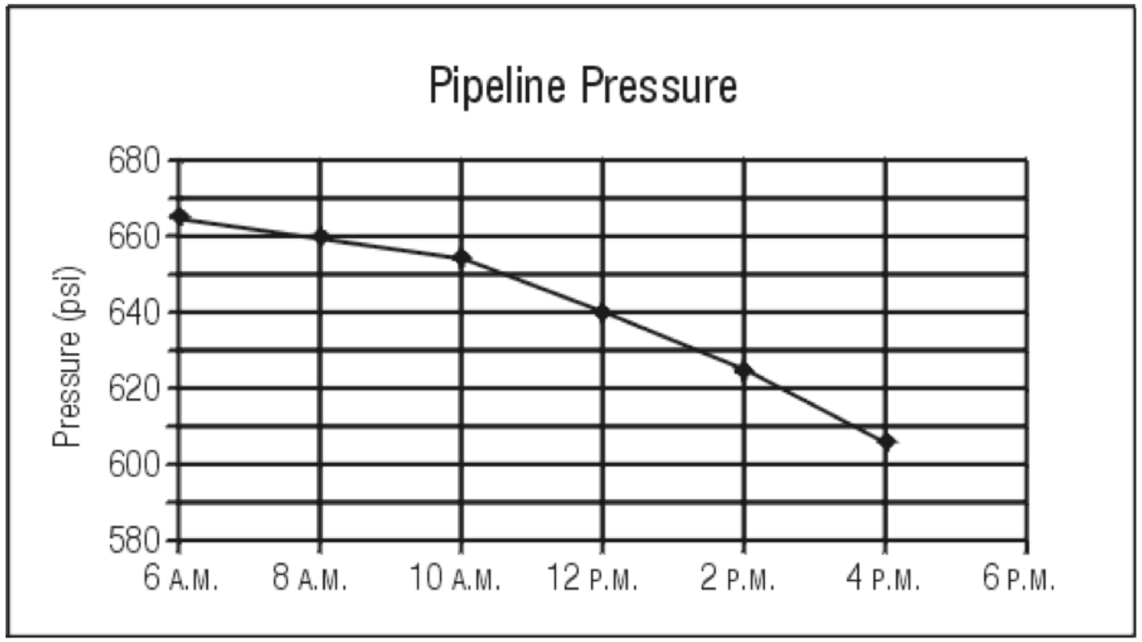
On a worksheet, trace the graph, leaving room on the right and above. Now use your ruler to extend the line graph on your paper beyond the value corresponding to 110°C on the horizontal axis. Then, using the procedure for reading the graph, find the temperature in degrees Fahrenheit that equals a temperature of 110°C. Do you get about 230°F?

In the graph, the horizontal and vertical axes were already labeled out far enough to allow you to extend the graphed line. Sometimes, you will have to extend these axes, too, so you can properly draw the rest of the graph and read it.

Now let’s consider an example of a line graph on which the relationship is better pictured by a curved line rather than a straight line.

Example 8

The graph below shows a line graph of how the pressure in a high-pressure pipeline changes from morning to afternoon on a certain day. Notice the label on the side of the graph. It tells you the values of pressure in “pounds per square inch” (psi) that are plotted on the graph. The data are plotted with dots connected with short line segments. Notice that the overall effect is a “curved line.” When you look at this graph, you can compare the afternoon value with the morning value and see that the pipeline pressure is getting lower.



You can also see that the pressure is dropping faster in the afternoon than in the morning. When you look at a line graph and notice how the values are changing, you are observing a trend or pattern. This is probably the most useful feature of line graphs, particularly graphs that use “time” on the bottom axis.

You can read values from the curved line in the graph, just as you did with the temperature-conversion graph. You may wonder what happened to the pressure between 10 a.m. and 12 p.m. The plotted points tell what the pressure was at 10 a.m. and at 12 p.m. Can you determine what the pressure was at 11 a.m.? As before, you must interpolate to find the answer. The curved line indicates that the pressure at 11 a.m. was somewhere between the values at 10 a.m. and 12 p.m., perhaps 648 psi. Does that seem reasonable?

Look again at the graph. Suppose you wanted to know what the pressure was going to be at 6 p.m. Can you extend this graph out to 6 p.m.? How confident would you feel about the value you determined by extending the graph?

Drawing Graphs

Drawing Graphs

You will probably have many occasions to draw line graphs. Follow these steps to draw a line graph:

First, decide what you are trying to show. Choose a title that describes your graph accurately.

Second, choose the general labels for the axes. It may be helpful to arrange your information in a table first, to help you see clearly what you want to graph.

Third, figure out what units you want for your graph. Draw your axes. Mark and label them with the units you've selected.

Fourth, by going over and up from the origin, plot the points for your graph.

Fifth, begin at your first point and draw a line through the remaining points that best represents the data you've drawn. The line may be straight or curved, depending on the relationship among the values you are graphing.

Ohm's law states that in a circuit the product of the current (I) and resistance (R) is equal to the applied voltage (V) or $V = I \times R$. Suppose currents and voltages were found as seen in table 7.1.

TABLE 7.1 Voltage and Current

Voltage (Volts)	Current (Amperes)
0	0
10	1
20	2
30	3
40	4
50	5

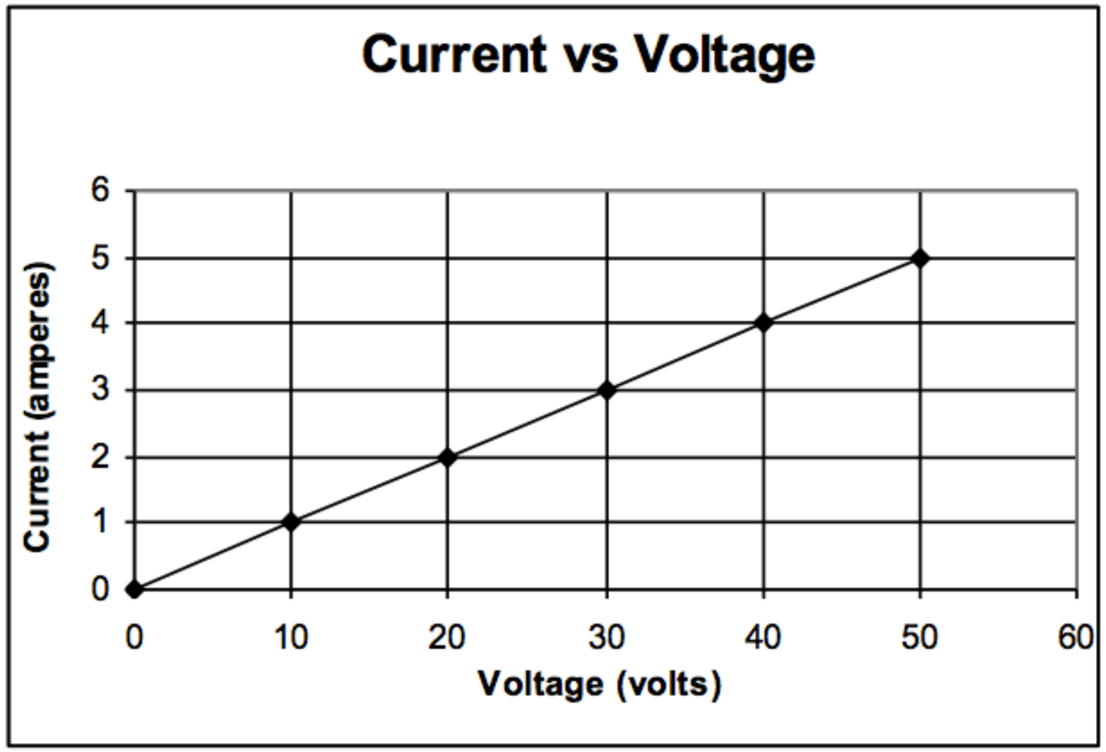
Look at the table of numbers you are going to plot to help you decide what title and labels to use for this graph. Next, draw the axes on a sheet of graph paper. Using regular spacing along the axes (you should be able to use the lines printed on the graph paper), mark the voltage along the x-axis (bottom); it is the independent variable. Then current goes on the y-axis (left) because it is the dependent variable (varies according to the independent variable). Another way to think of these terms is that the independent variable is the cause while the dependent variable is the effect.

Now you are ready to plot the points in the table. To plot the first point in the table (0 volts giving 0 amperes), begin at the origin and go 0 units over and then 0 units up. Of course you don't go anywhere! The origin is the first point on your line graph. Make a small dot at the origin to show that you have graphed these values as the first point.

To plot the second point in the table (10 volts giving 1 ampere), begin at the origin and go over to 10 along the horizontal axis. Then from there go 1 unit straight up. Make a small dot at that point. In the same way, plot the third, fourth, fifth, and sixth sets of data.

Now use a straightedge to draw a line that joins the first point to the second point, the second point to the third, and so on. This particular graph is a straight line. Is the line you drew a straight line (or fairly close)? If it isn't, plot the points carefully again.

Compare your graph to the one shown below and make any needed corrections.



You may have noticed that we have not mentioned the third variable of Ohm's law, resistance. Solving for resistance in Ohm's law would result in $R = \frac{V}{I}$. So, for any ordered pair on the graph (or table), we can find the resistance by dividing the voltage by the current (or the variable on the x-axis divided by the variable on the y-axis).

$$R = \frac{V}{I} = \frac{\text{voltage}}{\text{current}} = \frac{x}{y}$$

You may also recognize that $\frac{x}{y}$ is essentially the inverse of the slope $\left(\frac{\Delta y}{\Delta x}\right)$. Therefore, in this problem the resistance (R) is the inverse of the slope (m) of the graph.

$$m = \frac{\Delta y}{\Delta x} = \frac{1}{R}$$

Solution to Scenario Questions

Solution to Scenario Questions

Following are the solutions to the questions posed under “Photonics Scenario Involving Graphing in Rectangular Coordinates.”

a. From the curve for NaCl, it appears that no more than 40 g (the maximum amount of NaCl on the vertical axis) can be dissolved in 100 g of water, no matter how hot the water is (the horizontal axis).

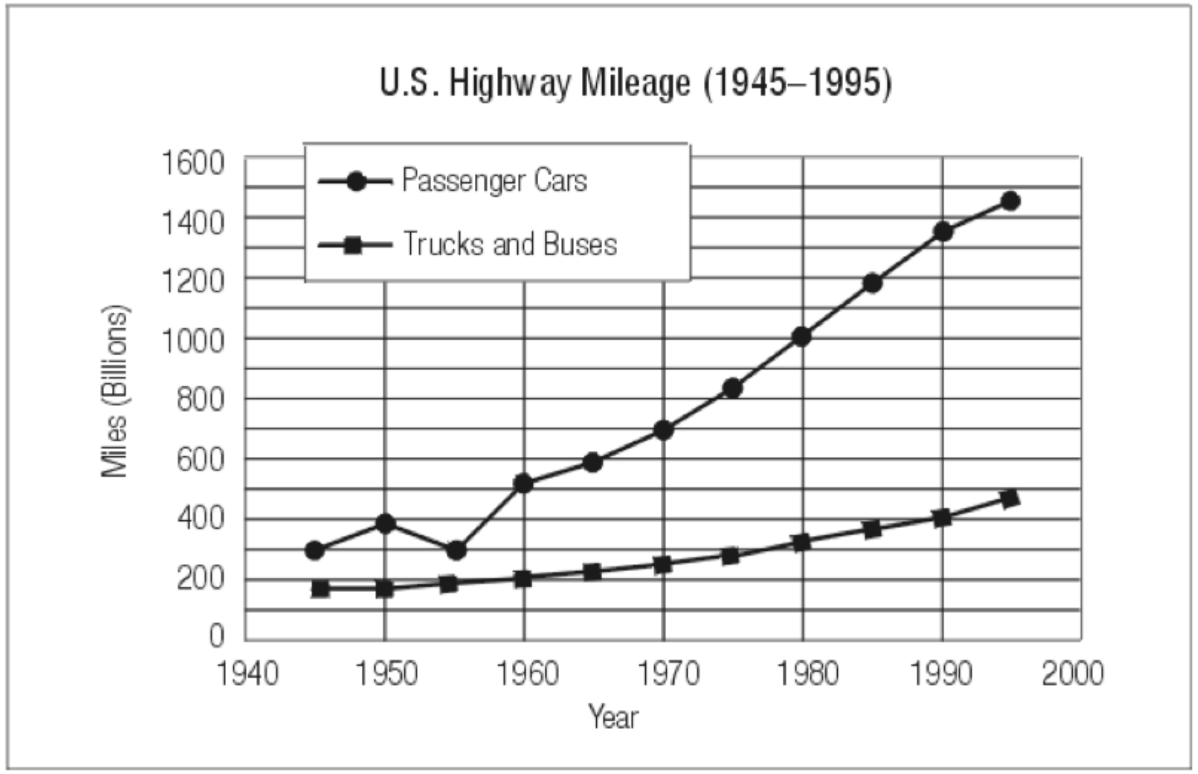
b. Yes, from the graph for KNO_3, it appears that 100 g of KNO_3 will dissolve in 100 g of water when the water temperature is about 57°C. At temperatures warmer than this, even more KNO_3 will dissolve.

c. Reading from the graph, follow the line for 70°C water (on the horizontal axis) up to the curve for NH_4Cl, and then read the corresponding amount over on the vertical axis: 60 g of NH_4Cl will dissolve in 70°C water.

Practice Exercises

Exercise 1

Information about the total highway mileage traveled by vehicles in the United States is shown below.



a. What types of vehicles are referred to by this graph?

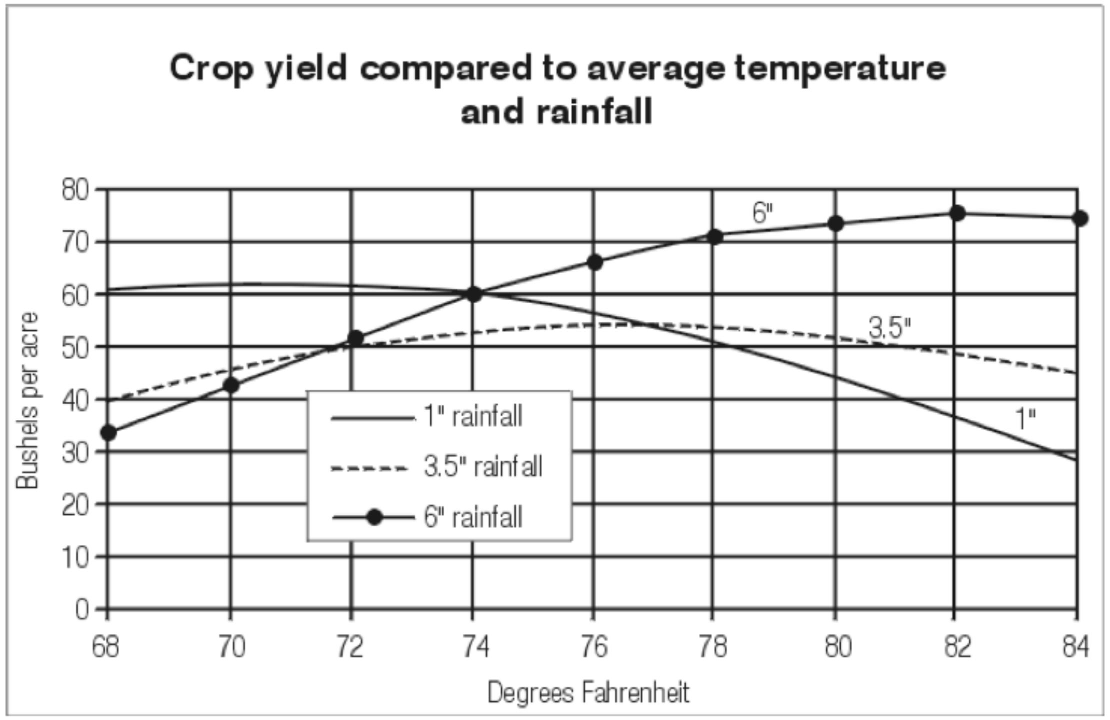
b. What span of time is covered by the graph?

c. The total mileage traveled each year is increasing. Which type of vehicle is increasing its total mileage faster?

d. Use the graph to estimate how many miles trucks and buses will travel in the year 2000. Do you think this would be a reliable estimate? Why?

Exercise 2

There are many relationships between weather and crop performance. A relationship pertaining to corn yields in a certain region is pictured below. It compares crop yield with average temperature for the month of July, for various rainfalls.



a. What units are being used for (1) crop yields, (2) average temperature, and (3) rainfall?

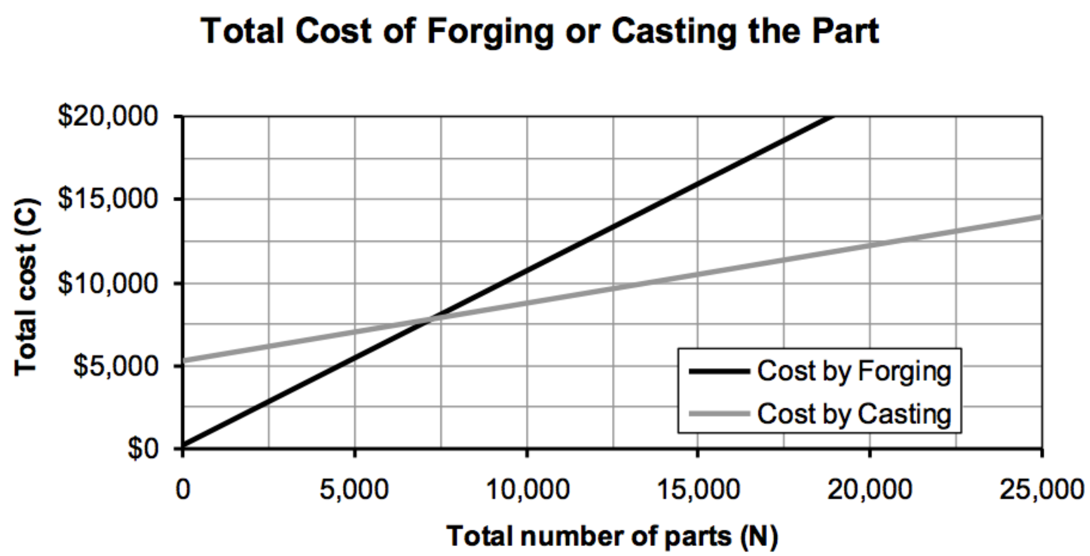
b. How is the graph drawn so that you can identify the lines for the three different rainfalls?

c. For an average temperature of 82°F, what crop yields can be expected for a 6" monthly rainfall? For a 1" monthly rainfall?

d. Which is better for a cool average temperature of 70°F—heavy rainfall or light rainfall?

Exercise 3

A large company produces a certain machine part. The company can produce the part either by casting or by forging. The tooling cost for casting is quite expensive, but the labor and material cost is relatively low compared to the forging method. With the forging method, the tooling cost is lower but the labor and material cost is higher. A graph of these relationships is as follows.



a. For an order of 15,000 parts, what is the cost of producing the part by casting? By forging?

b. For an order of 5000 parts, what is the cost of producing the part by casting? By forging?

c. At what size order, the breakpoint, are the cost of forging and the cost of casting the same?

d. Can the "breakpoint" determined above be used as a guideline for the production staff? What would that guideline be?

Exercise 4

Traveling at 50 miles per hour, you record the mileage on your odometer every 30 minutes, as shown below.

Time (min)	Distance traveled (mi)
0	0
30	25
60	50
90	75
120	100

a. Draw and label the axes for your graph. Use the labels and title in the table above to help you label the axes. Put the driving time along the x-axis (0, 30, 60, 90, and 120) and distances traveled along the y-axis (up to 100). Then plot the information from the table, using the driving time and the distance traveled. Plot each set of data given in the table and join the points on the graph with lines. Your graph should be a straight line. If your graph is not fairly straight, check your points and plot them again.

b. According to your graph, on your next trip (at the same speed) about how many miles can you travel in 50 minutes? How many in 100 minutes? Remember that you are interpolating with your graph; the result is only an estimate.

c. Suppose you wanted to show the bottom axis in hours, rather than minutes. Relabel your bottom axis in hours without replotting your points.

d. Now use your graph to tell how far you could travel in 3 hours. Do you think this is a reasonable extrapolation of your graph?

Exercise 5

As a lab technician, you must perform various tests of material properties. A stress analysis on wire samples is one such test. Shown below are the results of a stress analysis on a length of copper wire. The stretch of the wire is measured as different weights are hung from it.

Stress analysis on copper wire

Load (lb)	Stretch (inches)
0	0.00
2	0.02
4	0.04
6	0.06
8	0.08
10	0.10
12	0.13
14	0.64

a. Construct a graph of the stretch in the copper wire at each load. Label the load along one axis and the stretch along the other axis.

b. Interpret the graph to estimate the load limit for this wire. Show the load at which the wire will stretch to the point of breaking.

c. Can you estimate what the stretch might be at 3 pounds? At 16 pounds? If so, what is your estimate? If not, explain why you are not able to estimate.

Solutions to Practice Exercises

1. a. Passenger cars and trucks and buses

   b. 1945 to 1995

   c. Passenger cars

   d. About 550. This estimate should be fairly reliable because over the last 50 years the increase has been consistent.

2. a. (1) bushels per acre, (2) Fahrenheit, (3) inches

   b. Each line is formatted differently so the user can distinguish between them.

   c. 6" about 75 bushels per acre

      1" about 37 bushels per acre

   d. Light rainfall (1" rainfall results in about 62 bushels per acre.)

3. a. 15,000 parts casting about $10,500

      forging about $16,000

   b. 5000 parts casting about $7000

      forging about $5500

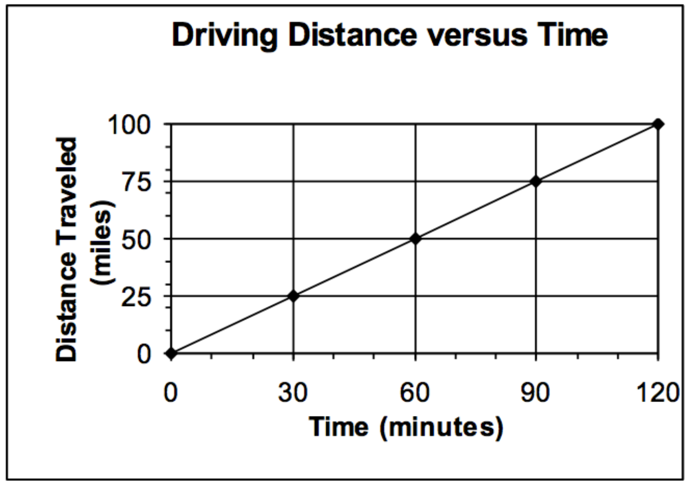
   c. About 7500 parts

d. Yes. If producing less than 7500 parts, it is more cost effective to use the forging method. If over 7500 parts are being produced, casting will save money.

Parts < 7500 ⟶ forging

Parts > 7500 ⟶ casting

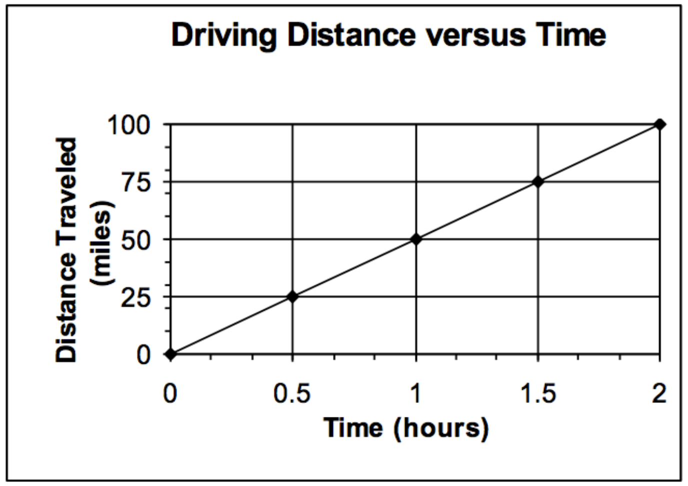
4. a.



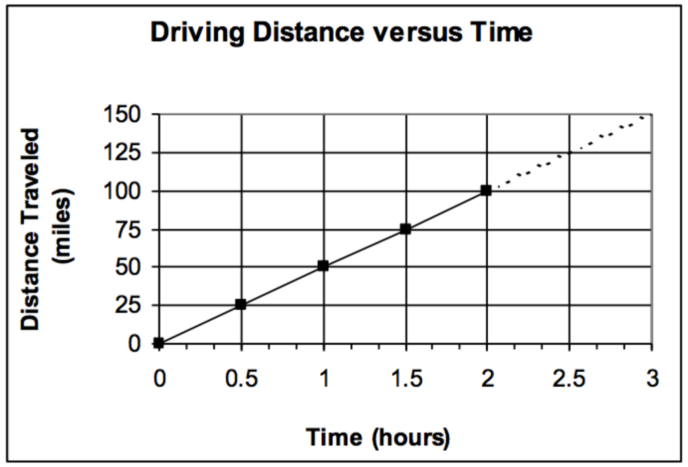
b. In 50 minutes you would travel about 42 miles.

In 100 minutes you would travel about 84 miles.

c.

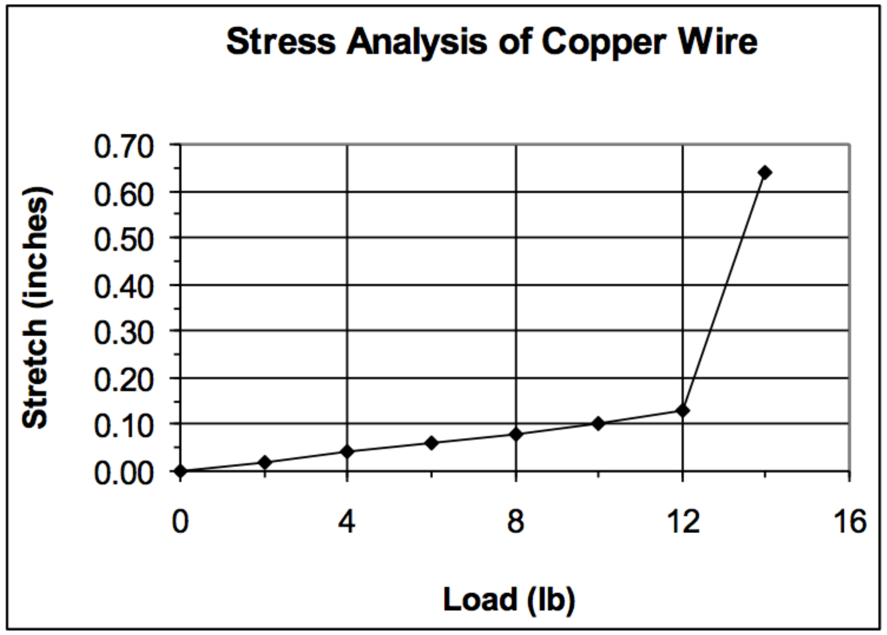


d.



150 miles; this is a reasonable extrapolation. However, the more time spent in the car, the greater the likelihood that you will stop for gas, food, etc.

5. a.



b. Somewhere between 12 and 14 pounds the stretch distance increases drastically.

c. 3 pounds about 0.03 inch

At 16 pounds an accurate estimation is not possible. We know that it would exceed 0.64 inch, but we know little beyond that. The problem is that there is not a consistent pattern at and around this point to project the stretched distance. The "point of breaking" in a material can be modeled with a piece of silly putty. Take a piece and pack it tightly together. Then slowly pull on both ends. You'll notice that initially there is resistance to the stretching. But once it gets to a certain point the resistance greatly decreases as you pull it apart.

Career Video

Adventurous Career Opportunities In Wind Energy

Wind turbine service technicians work outdoors and hundreds of feet in the air maintaining, troubleshooting and repairing wind turbines.

Geometry

8

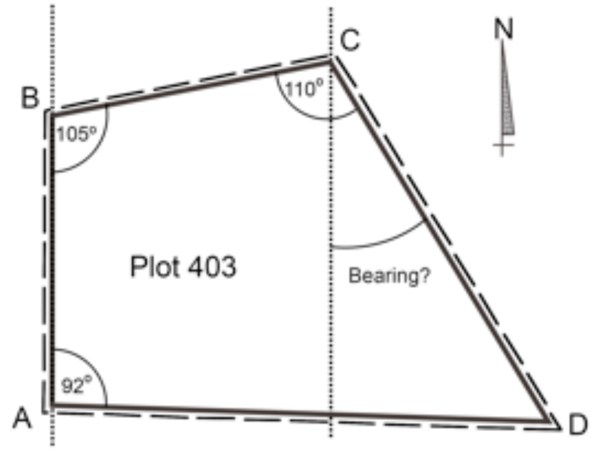
Objectives

When you have completed this activity, you should be able to do the following:

1. Recognize parallel and perpendicular lines and understand their properties
2. Measure angles
3. Find unknown angle measures in triangles algebraically

Scenario

A "closed traverse" of a certain plot of land shows the angles of traverse, but not the directional bearings (compass headings). Bearing angles are measured relative to North (so that North is 0°, East is 90°, and so forth). In the drawing shown here, the dashed lines are parallel north-south lines. What is the directional bearing of line $\overline{CD}$?

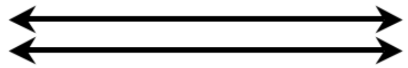


Before looking at the solutions, work through the lesson to further develop your skills in this area.

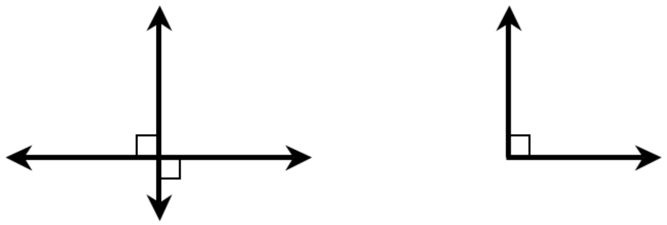
Lines

Lines

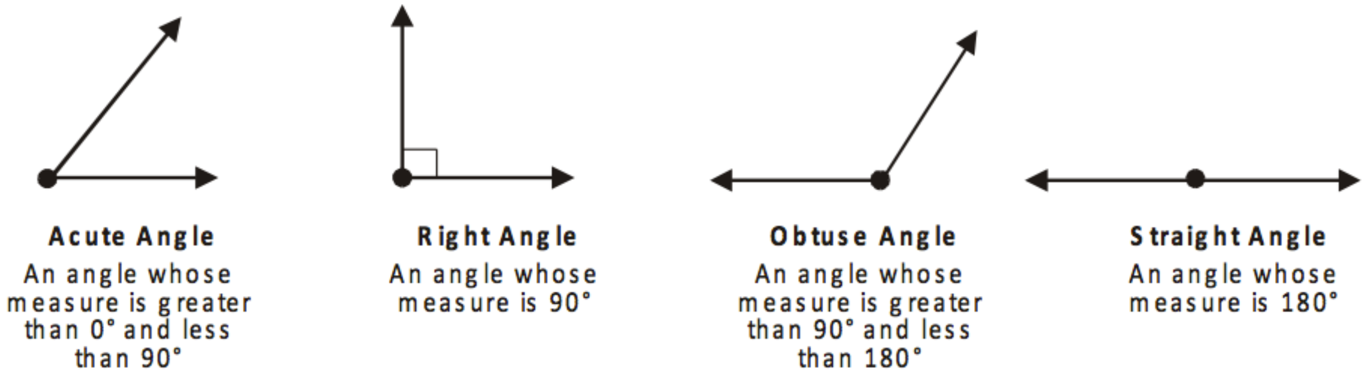
Parallel lines—Lines in the same plane that never intersect are parallel lines.



Perpendicular lines—Lines that intersect to form a right angle are perpendicular lines. Note the “square symbol” is used to identify a right angle.



Angles—An angle is the figure formed by two rays with a common endpoint. Angles may be classified by their measures.

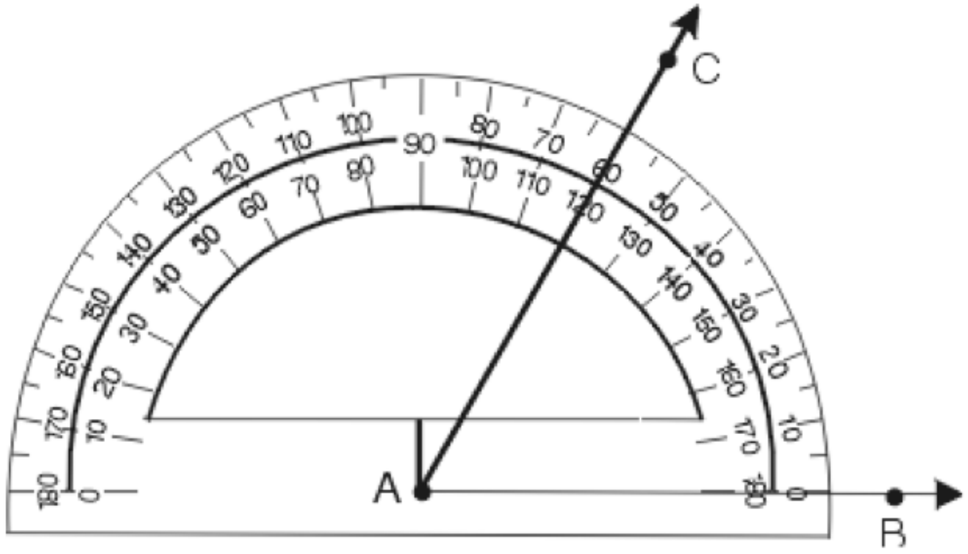


Angles are measured as parts of a circle. For measuring in degrees, the circle is divided into 360 parts. Each part is one degree, written 1°.

A **protractor** is a tool for measuring angles.

Example 1

Measure ∠CAB in the illustration below.



Solution

a. Place the midpoint mark of the protractor over vertex A of ∠CAB.

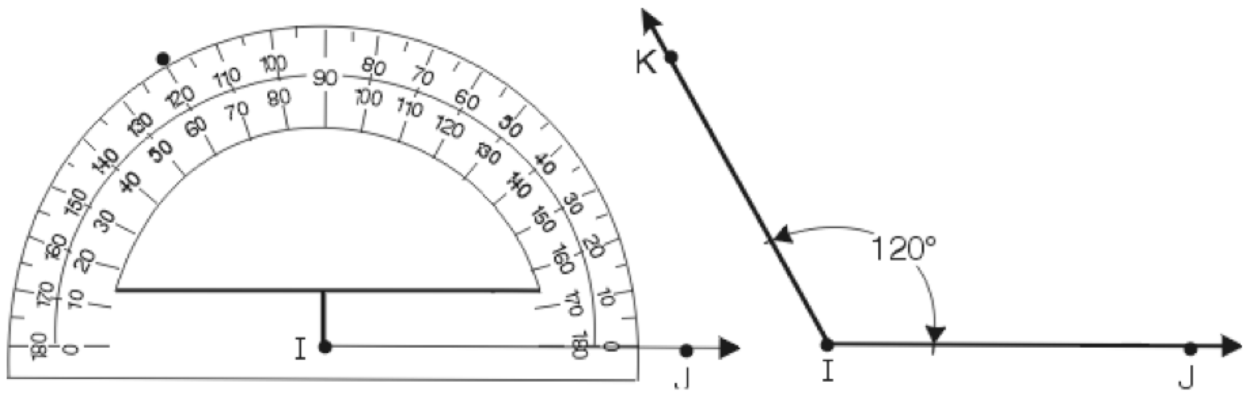
b. Align line with the 0° mark on the protractor. This example uses the outside scale.

c. The measure of ∠CAB is the number where the line crosses the scale of the protractor. The measure of ∠CAB is equal to 60°. You can write this as ∠CAB = 60°.

Example 2

Draw an angle of 120º.

Solution

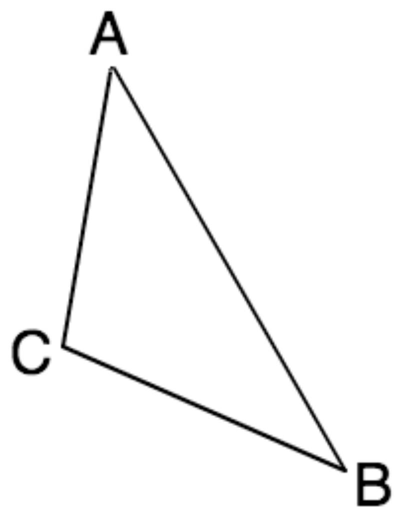


Draw a ray. Label the ray $\overrightarrow{IJ}$ Place the midpoint of the protractor over the endpoint I. Align $\overrightarrow{IJ}$ with the 0° mark on the outside scale. Make a point at 120°. Label the point K. Draw $\overrightarrow{IK}$.

Triangles

Triangles

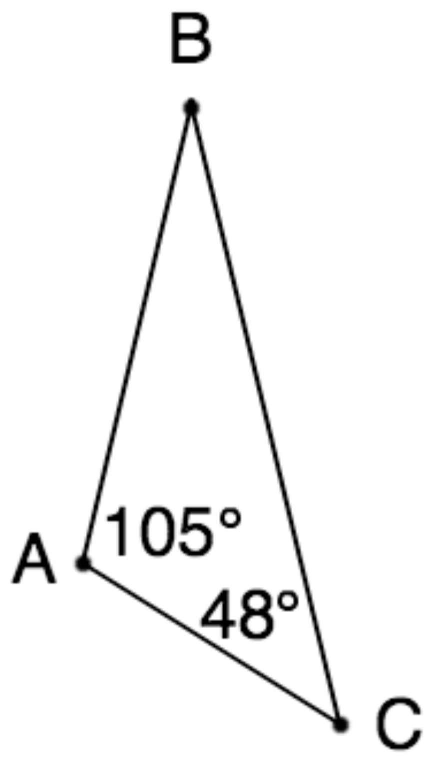
A triangle is a figure with three sides and three angles. The sum of the measures of the angles in a triangle is always equal to 180°.



$$\angle A + \angle B + \angle C = 180°$$

Example 3

Find the measure of ∠B in △ ABC here.



Solution

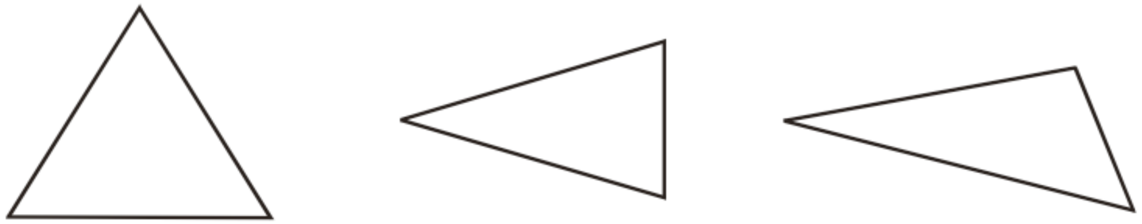
$\angle A + \angle B + \angle C = 180°$

$105° + \angle B + 48° = 180°$

$\angle B + 153° = 180°$

$\angle B = 27°$

Triangles are often classified by the number of sides that are congruent.



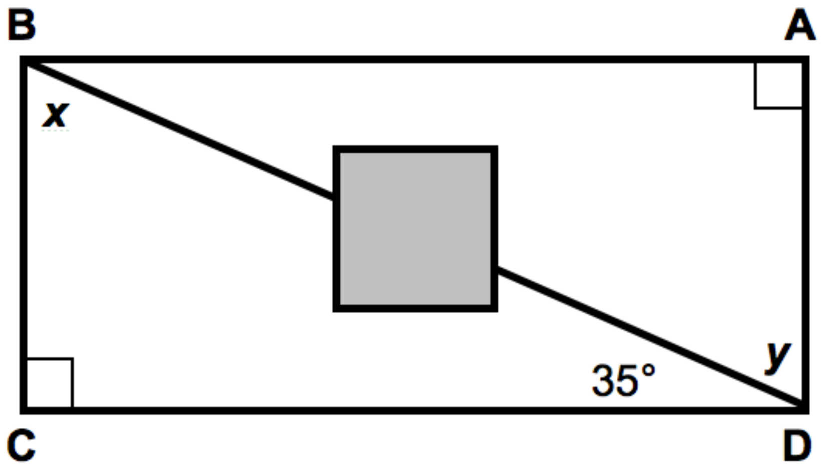
Equilateral 3 Congruent sides — Isosceles 2 Congruent sides — Scalene No Congruent sides

Triangles are also classified by their angles. Congruent angles are shown by the rounded marks on the angles below.

Equiangular 3 Congruent angles — Right 1 Angle measures 90° — Acute Each angle measures less than 90° — Obtuse 1 Angle measures more than 90°

Example 4

You are building a rectangular metal toolbox with a divider along the diagonal to keep tools sorted. The toolbox has a compartment in the middle for long narrow tools such as hammers, screwdrivers, and small saws. You determine that the angle of the divider at one corner of the box should be 35°. (a) What should be the measurement of $\angle x$ for the divider to be along the diagonal of the rectangle? (b) What then is the measure of angle $\angle y$? How does it relate to $\angle x$?



Solution

a. The sum of the angles of a triangle is 180°. A rectangle has four right angles. Therefore, the diagonal **BC** forms two right triangles, $\triangle BAD$ and $\triangle BCD$. Use the given angle and the right angle to find $\angle x$.

$$35° + \angle x + \angle C = 180°$$

$$35° + \angle x + 90° = 180°$$

$$\angle x + 125° = 180°$$

$$\angle x = 55°$$

b. As noted in part (a), a rectangle has four right angles. The originally given angle (35°) and $\angle y$ together make up one of these at $\angle D$. (We call them complementary angles.) Therefore we can make the following statement:

$$35° + \angle y = 90°$$

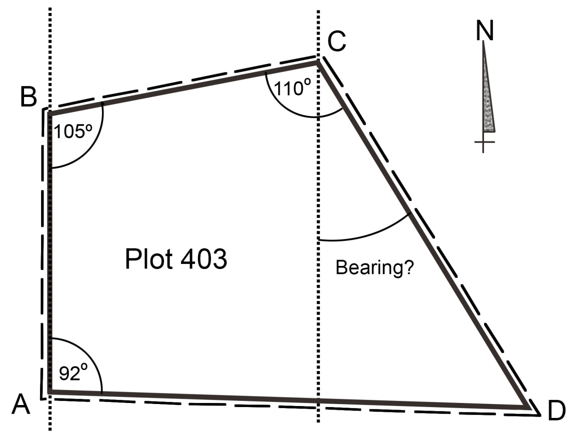
$$\angle y = 55°$$

Therefore, $\angle x = \angle y$

Solution to the Scenario Question

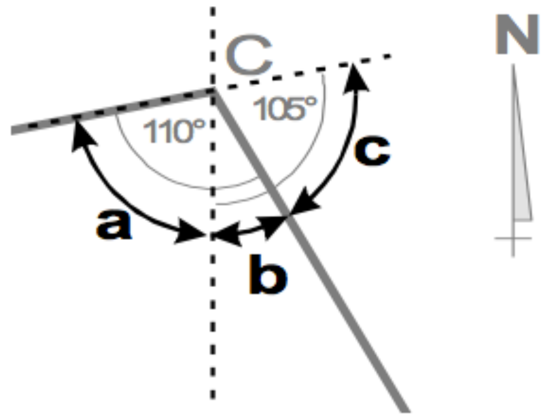
Solution to Scenario Question

Here is one way to approach this problem. When a transverse line intersects two parallel lines, the pairs of corresponding angles are equal. Thus, as line $\overline{BC}$ crosses the pair of parallel north-south lines, the corresponding angles made by $\overline{BC}$ and the north-south lines are 105°, as shown in the drawing to the right.



Next, use a little algebra! Since the sum of all the angles along $\overline{BC}$ must be 180°, we can write three algebraic statements involving the three angles labeled a, b, and c in the drawing shown here:

$180° = a + b + c$ $110° = a + b$ $105° = b + c$



where b is the bearing angle we are seeking. We have three equations with three variables. We can solve them by substitution.

First, solve the second equation for a:

$$a = 110° - b$$

Next, solve the third equation for c:

$$c = 105° - b$$

Finally, substitute these expressions for a and c into the first equation, and then solve for b.

$180° =$	$a + b + c$	The first equation (given).
$180° =$	$(110° - b) + b + (105° - b)$	

Substitute the expressions for a and c.

$180° =$	$215° - b$	Simplify.
$b =$	$35°$	Solve for b.

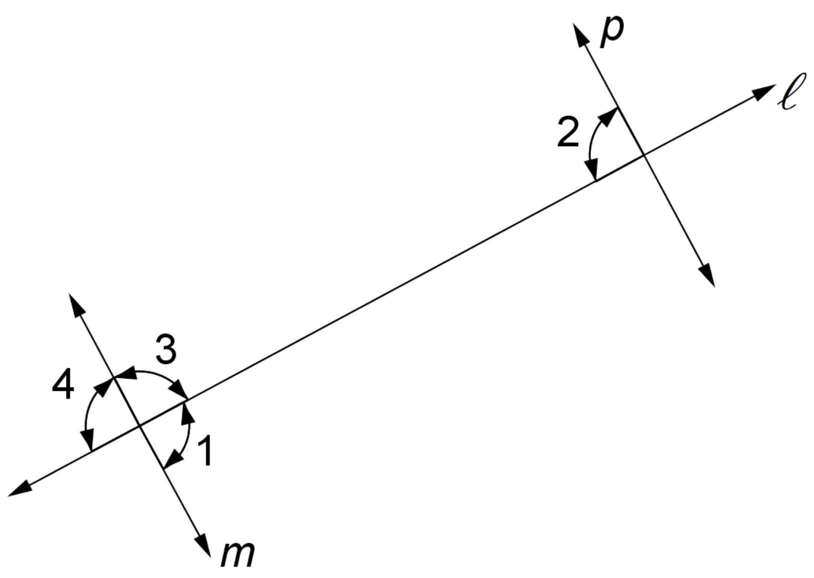
So, the bearing angle for $\overline{CD}$ is 35° east of south, or a heading of 145° (clockwise from North).

Practice Exercises

Practice Exercises

Exercise 1

Use the drawing below to answer the questions that follow:



a. If ∠1 = 90°, what can you say about lines ℓ and *m*?

b. If line *p* is perpendicular to line ℓ (p ⊥ ℓ), what is the measure of ∠2 ?

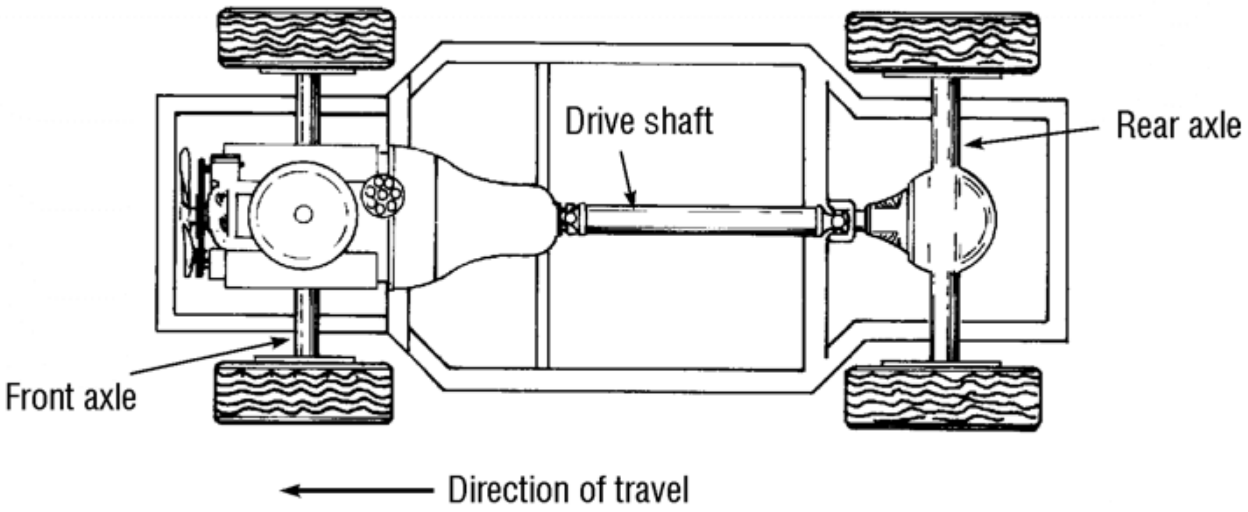
c. If ∠1 = 90°, what is the measure of ∠3 ?

d. If ∠1 = 90°, what can you say about the measure of ∠4 ?

e. Since ∠1 and 2 are right angles, what statement can you make about lines *m* and *p*?

Exercise 2

Below is a simple sketch of the frame and drive shaft for a rear-wheel drive automobile. Complete the statements that follow.



a. The drive shaft is ______ to the rear axle.

1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

b. The rear axle and the front axle are ______ to each other.

1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

c. The drive shaft is ______ to the direction of travel of the car.

1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

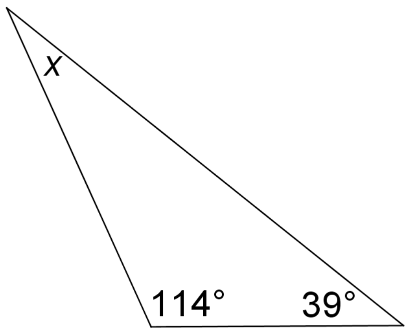
d. When the tires on the car have spun around 360° (one revolution), the distance they have moved on the ground is about the same as their ______.

1. radius
2. diameter
3. circumference
4. semicircle

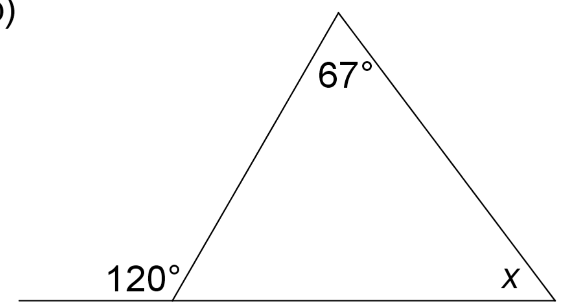
Exercise 3

For the following triangles solve for ∠x.

a)



b)



Solutions to Practice Exercises

Solutions to Practice Exercises

1. a. They are perpendicular $(\ell \perp m)$ because they intersect at a right angle (90°).

   b. $\angle 2 = 90°$; the reverse reasoning of part (a).

   c. $\angle 3 = 90°$; because 1 and 3 together make a straight angle (180°), so $\angle 1 + \angle 3 = 180° \Rightarrow 90° + \angle 3 = 180° \Rightarrow \angle 3 = 90°$.

   d. $\angle 4 = 90°$; if $\angle 1 = 90°$ then $\angle 3 = 90°$ (from part (c)), then since $\angle 3$ and $\angle 4$ make up a straight angle then $\angle 4 = 90°$.

   e. Lines *m* and *p* are parallel ($m \parallel p$) because each line is perpendicular ($\perp$) to the same line, ℓ.

2. a. 2

   b. 1

   c. 1

   d. 3

3. a. $114° + 39° + x = 180°$

$153° + x = 180°$

$x = 27°$

b. $(180° – 120°) + 67° + x = 180°$

$60° + 67° + x = 180°$

$127° + x = 180°$

$x = 53°$

Career Video

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At GE Aviation and Land O'Frost, the work of multi-skilled technicians extends beyond the assembly line to supporting production, maintaining equipment, troubleshooting computers and much more.

Angle Measures in Two and Three Dimensions

9

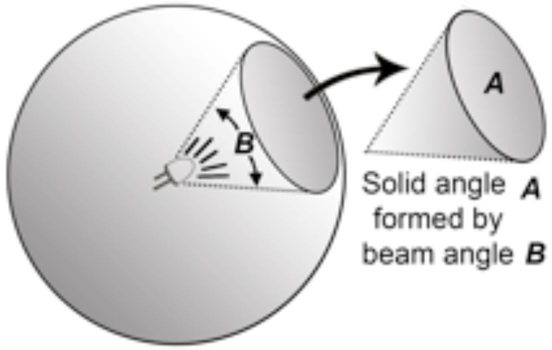
Objectives

When you have completed this section, you should be able to do the following:

1. Convert back and forth from degrees to radians (two dimensions)
2. Calculate partial solid angles (three dimensions)

Scenario

The brightness of small lights, such as LEDs, are typically reported in candela (cd) units, or millicandela (mcd). A candela is the intensity of a light source that produces 1 lumen of illumination per steradian of solid angle. The solid angle illuminated by a light source with a beam width of some angle B is:



$$\text{Solid Angle A} = 2\pi(1 - \cos \tfrac{B}{2}) \text{ steradians}$$

A certain LED is advertised as producing a brightness of 2300 mcd with a beam angle of 15°. Another LED produces 740 mcd at a beam angle of 45°. Which of these LED's actually produces more illumination, that is, more lumens of output?

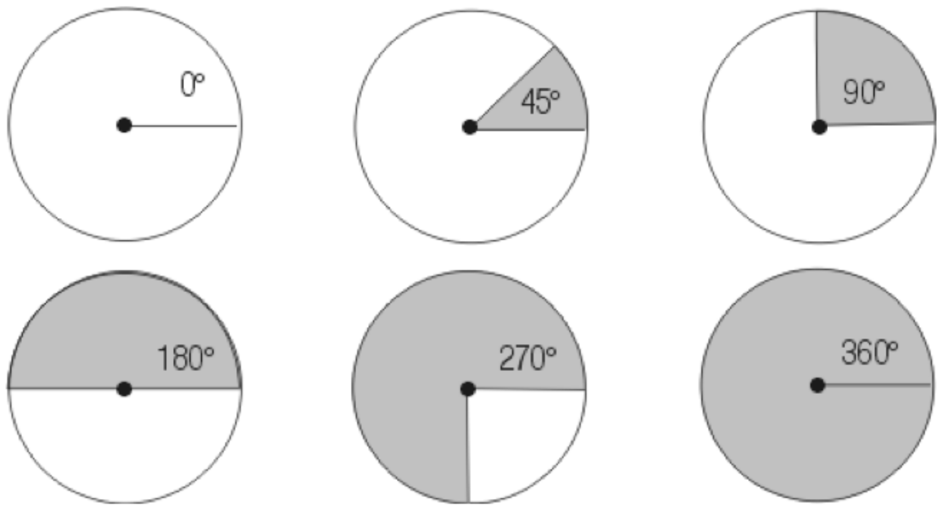
Hint: If a 1-candela point source radiated its light in every direction (a spherical pattern), it would have an illumination of 4π lumens, or about 12.6 lumens. That is,

Luminosity (in lumens) = Solid Angle (in steradians) × Brightness (in candela)

Before looking at the solution, work through the lesson to further develop your skills in this area.

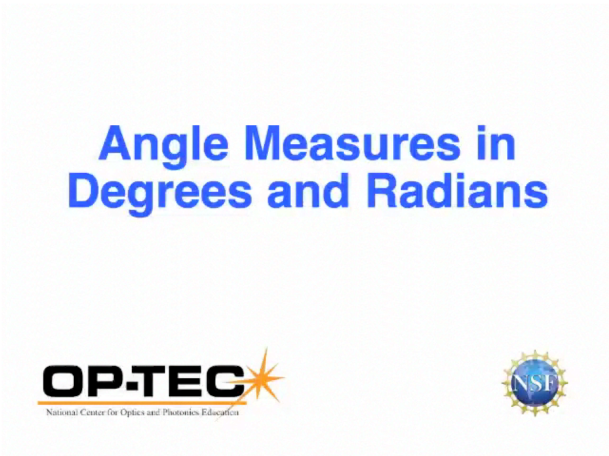
Radians and Degrees

In the previous section, we dealt extensively with angles and their measures in triangles. This section will explore the unique characteristics of angle measurement in two and three dimensions. Recall the following angle measures in circles, as shown here.



These measures are in degrees. Another common unit of angle measure (particularly in circles) is radians. Just as there are 12 inches in a foot, there are 180° in π radians or 360° in 2π radians. This makes for a rather simple conversion as seen in the following examples.

Movie 9.1 Angle Measures in Degrees and Radians



Click or Tap to Watch

Example 1

Convert 45° to radians.

$$45\cancel{^{\circ}} \times \frac{\pi \text{ radians}}{180\cancel{^{\circ}}} = \frac{\pi}{4}\text{radians}$$

Often the word radians is not included but is assumed. If "°" is used, it implies degrees. If there is no degree sign, radians should be assumed. When working with radians, it is customary to work with fractions of π, such as $\frac{\pi}{4}$ above.

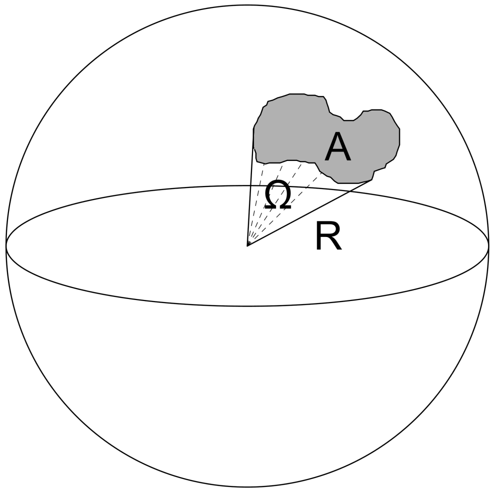
Example 2

Convert $\frac{3\pi}{2}$ to degrees.

$$\frac{3\,\pi}{2}\ \cancel{\text{radians}} \times \frac{180^{\circ}}{\pi\ \cancel{\text{radians}}} = 270^{\circ}$$

Solid Geometry

In solid geometry, the term "solid angle" is introduced by analogy with the idea of a plane angle in plane geometry. Thus, instead of two lines enclosing a plane angle, there now will be a conical surface enclosing a space. The enclosed space is called a *solid angle*. It is measured in *steradians* (abbreviated sr) and is usually denoted by the Greek letter omega (Ω). In plane geometry, 2π radians surround a point. In solid geometry, 4π steradians surround a point.



$$\Omega = \frac{\text{area intercepted}}{(\text{radius})^2}$$

$$\Omega = \frac{A}{R^2} \text{ in steradians}$$

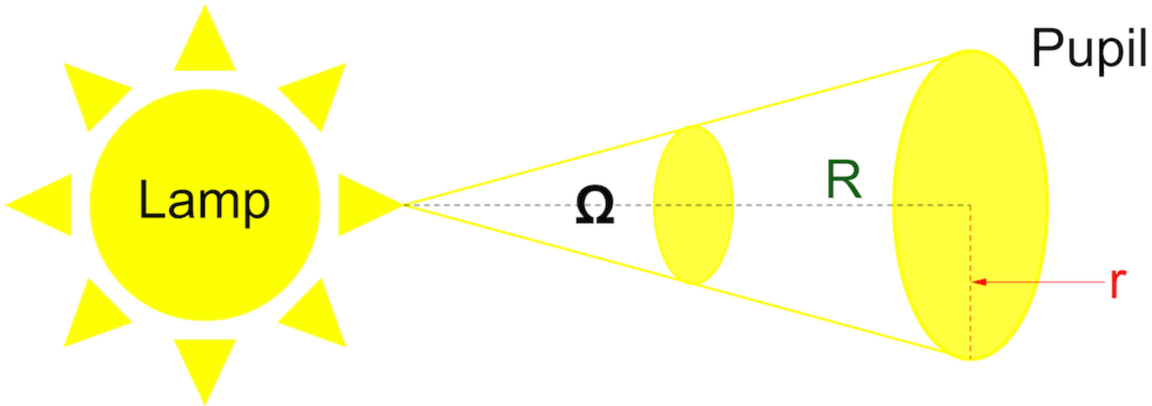
$$\Omega \text{ sphere} = \frac{4\pi R^2}{R^2} \quad \leftarrow \text{(surface area of a sphere)}$$

$$\Omega \text{ sphere} = 4\pi \text{ steradians}$$

The above figure shows a solid angle, Ω , defined in terms of the intercepted area on the spherical surface, A, and the radius of the sphere, R.

Example 3

Calculate the solid angle, Ω , that would be intercepted by the pupil of a man's eye (7-mm radius) from a lamp 2 meters away.



Solution

$$2\,\cancel{m} \times \frac{1000\text{ mm}}{1\,\cancel{m}} = 2000\text{ mm}$$

$$\Omega = \frac{\pi r^2}{R^2} = \frac{\text{(area of pupil)}}{(\text{distance from lamp to pupil})^2}$$

$$\Omega = \frac{\pi (7\text{ mm})^2}{(2000\text{ mm})^2}$$

$$\Omega = 3.85 \times 10^{-5}\text{ steradians}$$

Solution to Scenario Question

Solution to Scenario Question

The LED's light does not shine out in every direction. It radiates out only over a small portion of a sphere, as indicated by the advertised beam angle. We can use the formula given in the problem to calculate what solid angle is represented by each beam width, and then calculate the corresponding illumination (in lumens).

LED 1: Luminosity = (Solid Angle) Brightness

$$\text{Luminosity} = 2\pi\left(1 - \cos\frac{15°}{2}\right) \times 2300\,\text{mcd} \times \frac{1\,\text{cd}}{1000\,\text{mcd}}$$

Luminosity = 0.12 lumens

LED 2:

$$\text{Luminosity} = 2\pi\left(1 - \cos\frac{45°}{2}\right) \times 740\,\text{mcd} \times \frac{1\,\text{cd}}{1000\,\text{mcd}}$$

Luminosity = 0.35 lumens

So, the second LED, although with a notably lower candela brightness, actually generates more total illumination.

Practice Exercises

Exercise 1

Answer the following questions about angles and measuring angles.

a. How many degrees are in a circle?

b. If a pie is cut in half (across a diameter), then each of those halves is cut in half, and then each of those pieces is cut in half, how many pieces of pie are cut? How many degrees does each piece span?

c. How many degrees are between each of the numbers on the face of a clock? (That is, how many degrees are between the 12 and the 1, between the 1 and the 2, and so on?)

d. On the face of a clock, if you start at 12 o'clock and move clockwise, how many degrees will you cross between 12 and 3? Between 12 and 6? Between 12 and 9? Between 12 and all the way around to 12 again?

Exercise 2

Convert 60° to radians.

Exercise 3

Convert 150° to radians.

Exercise 4

Convert $\frac{\pi}{3}$ to degrees.

Exercise 5

Convert $\frac{5\pi}{4}$ to degrees.

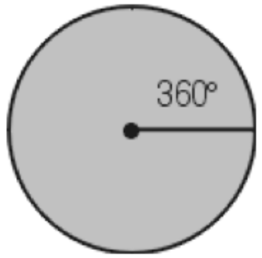
Exercise 6

Calculate the solid angle, Ω , that would be intercepted by a rectangular mirror (4 cm by 6 cm) from a lamp 1.5 meters away.

Solutions to Practice Exercises

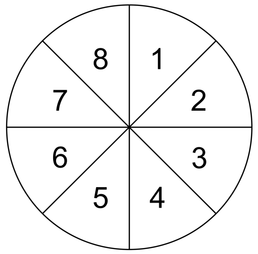
Solutions to Practice Exercises

1. a. 360°



b. 8 sections (equal size)

$$\frac{360°}{8} = 45° \text{ per piece}$$



c. There are 12 sections to a clock. So,

$$\frac{360°}{12} = 30°$$

d. 12 and 3

3 sections $3 \times 30° = 90°$ (forms a right angle)

12 and 6

6 sections $6 \times 30° = 180°$ (forms a straight angle)

12 and 9

9 sections $9 \times 30° = 270°$

12 to 12

12 sections $12 \times 30° = 360°$ (forms a full circle)

2. $45^{\circ} \times \frac{\pi \text{ radians}}{180^{\circ}} = \frac{\pi}{4}$ radians

3. $150^{\circ} \times \frac{\pi \text{ radians}}{180^{\circ}} = \frac{5\pi}{6}$ radians

4. $\frac{\pi}{3} \text{ radians} \times \frac{180^{\circ}}{\pi \text{ radians}} = 60^{\circ}$

5. $\frac{5\pi}{4} \text{ radians} \times \frac{180^{\circ}}{\pi \text{ radians}} = 225^{\circ}$

6. $1.5 \text{ m} \times \frac{100 \text{ cm}}{1 \text{ m}} = 150 \text{ cm}$

$$\Omega = \frac{l \times w}{R^2} \text{ steradians}$$

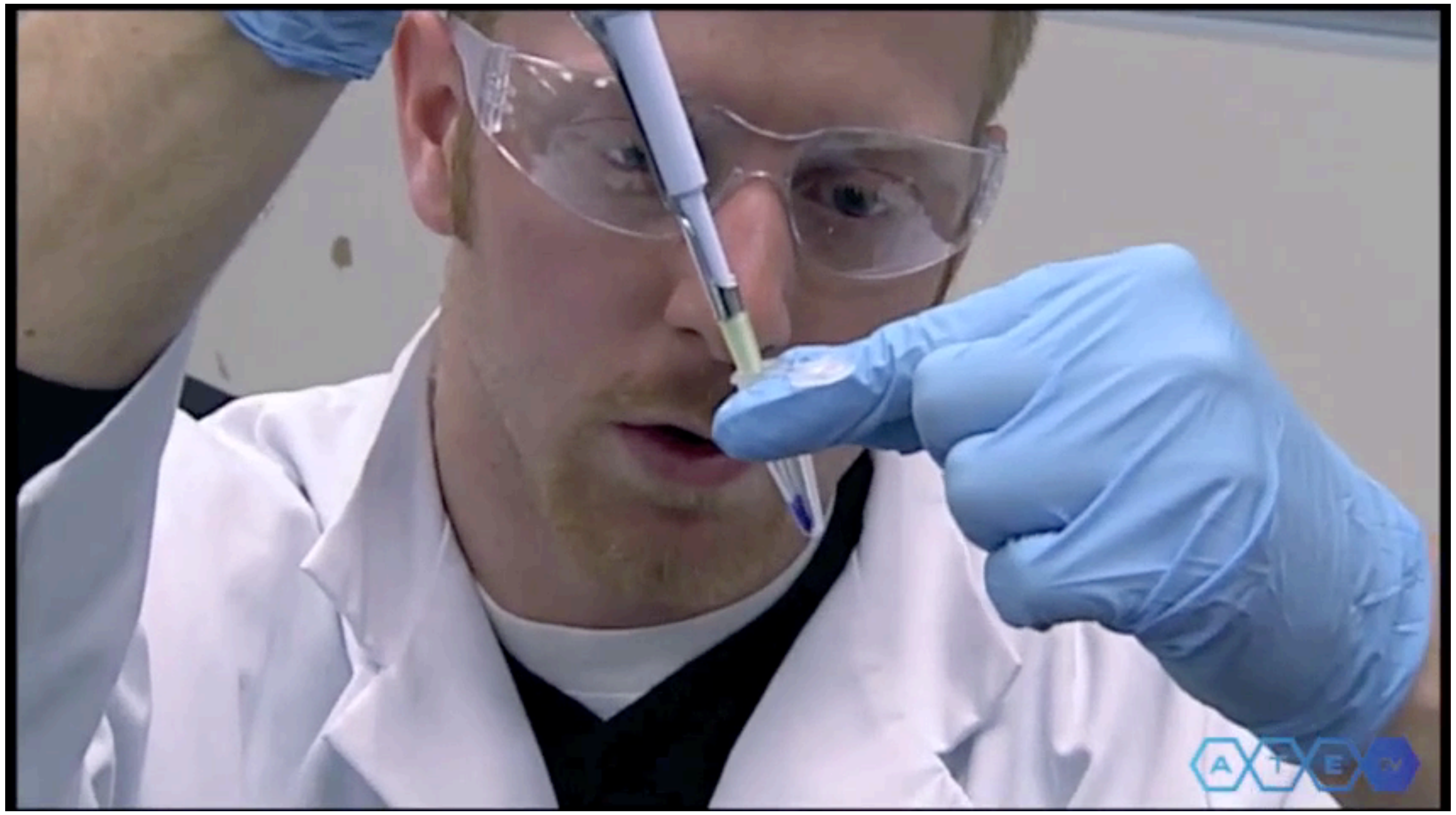
$$\Omega = \frac{4 \text{ cm} \times 6 \text{ cm}}{(150 \text{ cm})^2} \text{ steradians}$$

$$\Omega = \frac{24 \text{ cm}^2}{22500 \text{ cm}^2} \text{ steradians}$$

$$\Omega = 1.07 \times 10^{-3} \text{ steradians}$$

Career Video

On A Path To A Career In Lab



Visit a Research Methods in Molecular Biology Class where students learn that a "library" can contain something other than just books.

Trigonometry

10

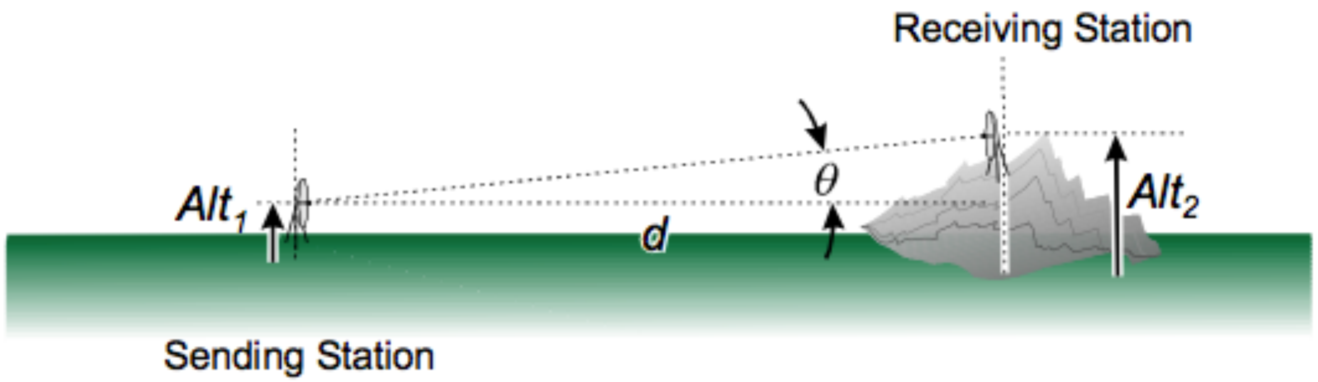
Objectives

When you have completed this section, you should be able to do the following:

1. Apply the Pythagorean theorem to solve problems
2. Use trigonometry functions (and their inverses) and know when to apply them

Scenario

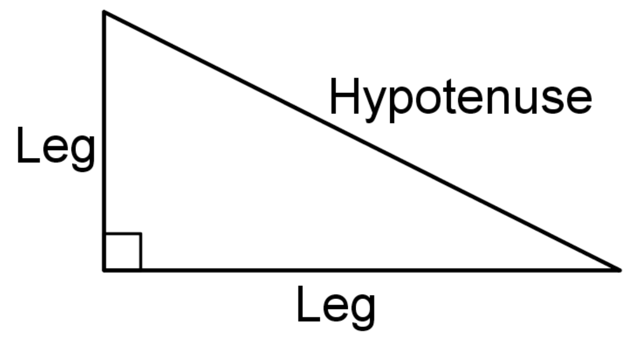
A microwave transmitter antenna mounted at 460 feet above sea level must be precisely aimed at a receiving antenna located 19.4 miles away, mounted on a mountaintop relay station at an altitude of 5,720 feet above sea level. What angle of elevation for the sending antenna will be required? What effect would the curvature of the earth have on the resulting angle?



Before looking at the solutions, work through the lesson to further develop your skills in this area.

Key Trigonometry Concepts

A **right triangle** is a triangle that contains a right angle. The **legs** of the triangle form the right angle. The side opposite the right angle is the **hypotenuse**. The legs and the hypotenuse are related by the Pythagorean theorem.

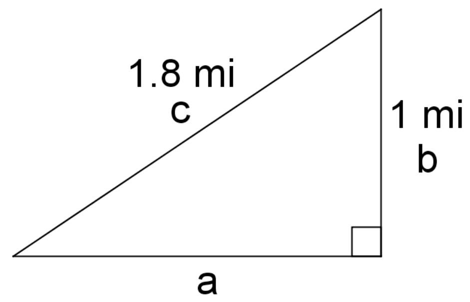


$$(\text{Leg1})^2 + (\text{Leg2})^2 = (\text{Hypotenuse})^2$$

Example 1

You reset your trip odometer at the bottom of a hill and then proceed to drive up the hill until you come to a sign that says you are 1 mile high. Your trip odometer reading says you have driven 1.8 miles. How far horizontally are you from your initial starting point?

It often helps to draw a picture to represent the problem. Here the hypotenuse, c, represents the ground.



Use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$

$$a^2 + (1.0\,\text{mi})^2 = (1.8\,\text{mi})^2$$

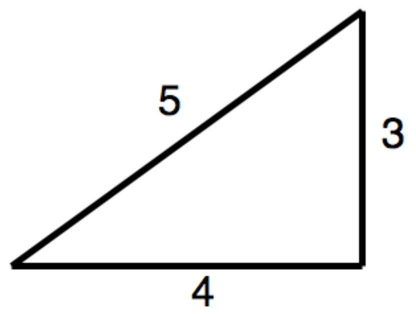
$$a^2 + 1.0\,\text{mi}^2 = 3.24\,\text{mi}^2$$

$$a^2 = 2.24\,\text{mi}^2$$

$$a = \sqrt{2.24\,\text{mi}^2} = 1.5\,\text{mi (rounded)}$$

You are 1.5 miles horizontally from your starting point.

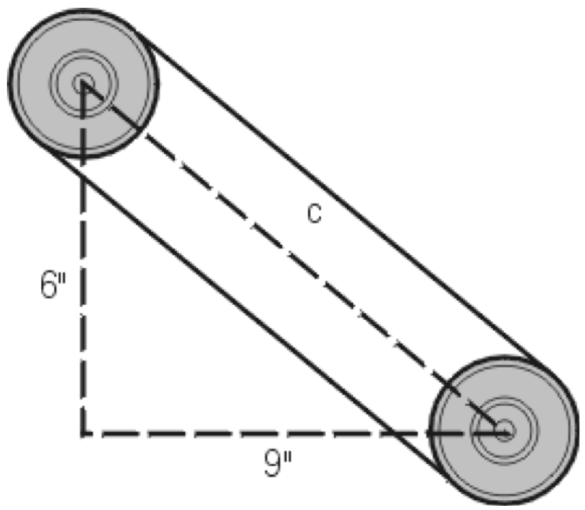
The whole numbers a, b, and c that satisfy $a^2 + b^2 = c^2$ are called a Pythagorean triple. For example, (3, 4, 5) is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Triangles with sides having lengths equal to Pythagorean triples are useful for solving problems.



Can you think of another Pythagorean triple?

Example 2

What is the distance between the centers of two pulleys if one is placed 9.0" to the left and 6.0" above the other.



Solution

$$a^2 + b^2 = c^2$$

$$(6.0 \text{ in})^2 + (9.0 \text{ in})^2 = c^2$$

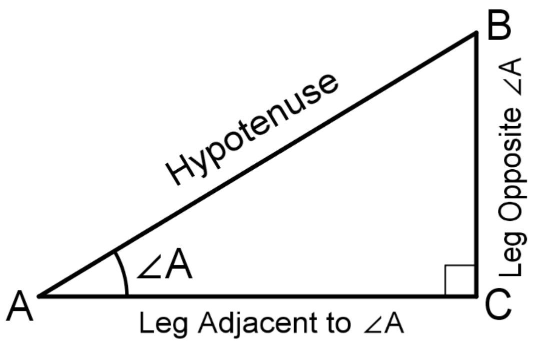
$$36 \text{ in}^2 + 81 \text{ in}^2 = c^2$$

$$117 \text{ in}^2 = c^2$$

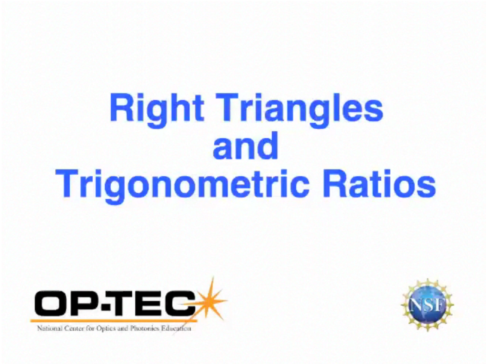
$$\sqrt{117 \text{ in}^2} = c^2$$

$$11 \text{ in (rounded)} = c^2$$

The measurements of the angles and sides of right triangles are related through the ratios of **trigonometry**. As shown in the illustration here, an acute angle of a right triangle has an adjacent leg and an opposite leg.



Movie 10.1 Right Triangles and Trigonometric Ratios



Click or Tap to Watch

The tangent, sine, and cosine ratios for this angle are defined as follows:

$$\text{Tangent of } \angle A = \frac{\text{Length of leg opposite } \angle A}{\text{Length of leg adjacent to } \angle A}$$

$$\tan A = \frac{\text{opp}}{\text{adj}}$$

$$\text{Sine of } \angle A = \frac{\text{Length of leg opposite } \angle A}{\text{Length of hypotenuse}}$$

$$\sin A = \frac{\text{opp}}{\text{hyp}}$$

$$\text{Cosine of } \angle A = \frac{\text{Length of leg adjacent to } \angle A}{\text{Length of hypotenuse}}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}}$$

Example 3

You need to determine the angle of a car's rear window with the vertical so you can cut a brake-light shield to match the angle. You measure the depth of the rear ledge as 12 inches and its corresponding height as 1012 inches.

Movie 10.2 Inverse Tangent

Click or Tap to Watch

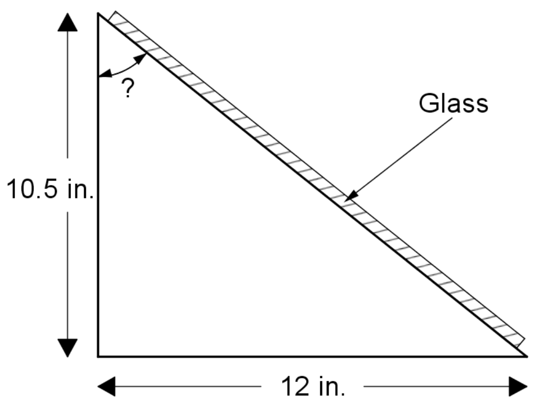
a. Make a sketch of the right triangle formed by the rear window. Label the known dimensions. Identify the angle that the window surface makes with the vertical.

b. Compute the value of the tangent of this angle by using the ratio of the lengths of the opposite and adjacent legs.

c. When the value of the tangent of an angle is known, the inverse tangent function (on your calculator) will give the value of the angle. Simply enter the value of the tangent (that is, the ratio) into your calculator and press the tan–1 key OR the INV key followed by the TAN key—the angle will be in the display. Do this to find the value of the angle for your rear window.

Solution

a. The sketch should appear generally as shown below.



b. The tangent of the angle is the ratio of the length of the opposite leg to that of the adjacent leg. It is important that you correctly identify the angle and the corresponding "opposite" and "adjacent" legs.

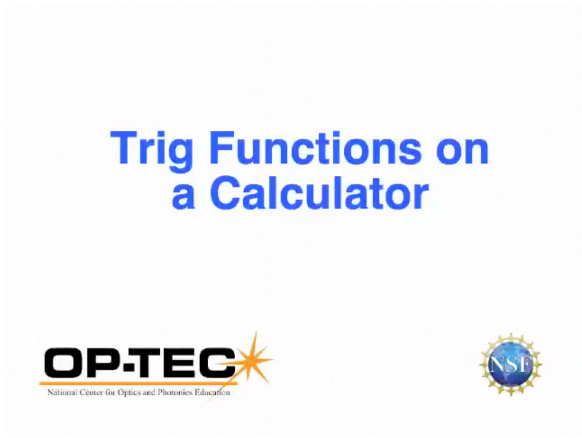
$$\text{Tangent of angle} = \frac{\text{Opposite leg}}{\text{Adjacent leg}}$$

$$\text{Tangent of angle} = \frac{12.0\text{ in}}{10.5\text{ in}}$$

$$\text{Tangent of angle} = 1.14286$$

c. Glass angle = INV TAN (tangent of angle)
Glass angle = INV TAN (1.14286)
Glass angle = 48.8° (rounded)

Movie 10.3 Trig Functions on a Calculator

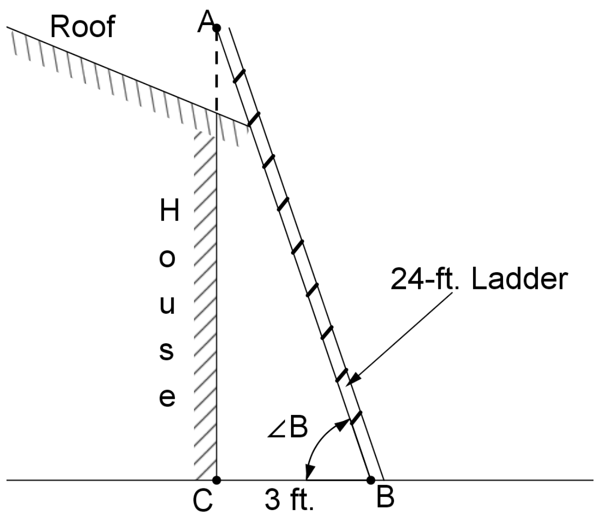


Click or Tap to Watch

Note: Some calculators must be set to "degree mode" to yield angles in degrees, rather than radians, for example. Also, the inverse tangent function may be labeled "$\tan^{-1}$."

Example 4

For safety purposes, a ladder manufacturer recommends that the angle a ladder makes with the ground should be greater than 65° but less than 80°. To repair a leak near the edge of the roof on your house, you need to extend a 24-ft ladder to its full length. When you do this, the base of the ladder is 3 ft from the vertical wall, as shown in the illustration. Can you safely use this ladder to make the repair? What are the minimum and maximum base distances for safe ladder use?



Solution

You need to find whether $65° < \angle B < 80°$. For the right triangle ACB, you know the lengths of the hypotenuse and the side adjacent to $\angle B$. Therefore, use the cosine ratio.

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos B = \frac{3}{24}$$

$$\cos B = 0.125$$

To find $\angle B$, use the inverse cosine ($\cos^{-1}$). You can use your calculator.

0.125 2nd $\cos^{-1}$ = 82.81924422

Since $\angle B$ 80°, you **cannot** safely use the ladder in this position.

To find the minimum and maximum base distances, again use the cosine relationship. Solve for the adjacent length using the minimum and maximum values of angle B.

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos 65° = \frac{\text{adj}}{24 \text{ ft}}$$

$$\text{adj} = 24 \text{ ft} \cdot \cos 65°$$

adj = 10.14 ft, or 10 ft (rounded)

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos 80° = \frac{\text{adj}}{24 \text{ ft}}$$

$$\text{adj} = 24 \text{ ft} \cdot \cos 80°$$

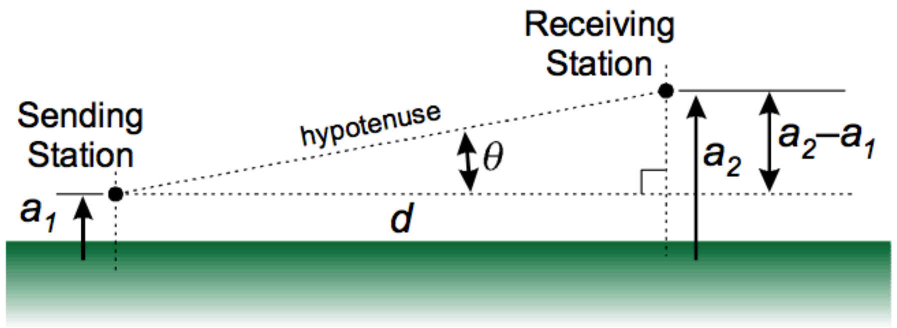
adj = 4.168 ft, or 4.2 ft (rounded)

The base of the fully extended ladder should be no closer to the wall than 4.2 ft and no farther from the wall than 10 ft.

Solution to Scenario Question

Solution to Scenario Question

Assuming the earth is flat, the problem is a straightforward application of the trigonometry ratios for a right triangle, simplified in the drawing here. (We're also assuming all the distances used below are in the same units.



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{(a_2 - a_1)}{d}$$

So, solving for …

$$\theta = \tan^{-1}\left(\frac{a_2 - a_1}{d}\right)$$

For the given problem, a_1 = 460 ft, a_2 = 5,720 ft, and
d = 19.4 mi 5280 ftmi = 102,000 ft. So,

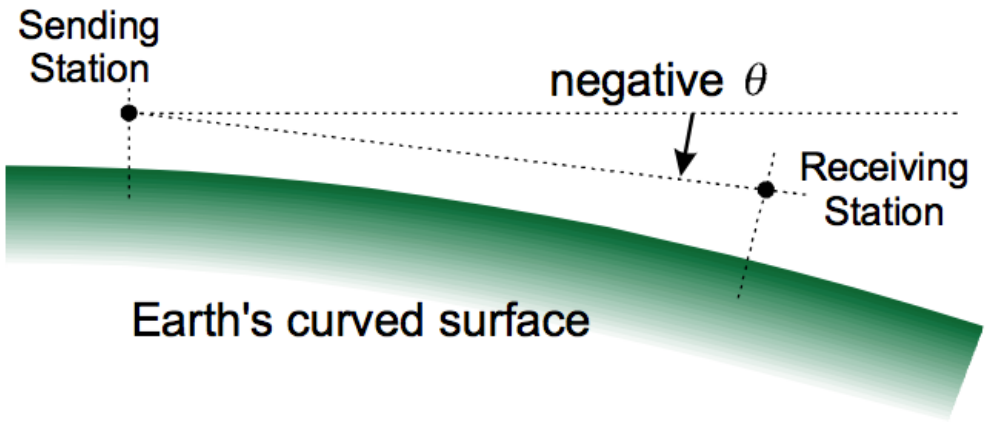
$$\theta = \tan^{-1}\left(\frac{5720\,\cancel{ft} - 460\,\cancel{ft}}{102{,}000\,\cancel{ft}}\right)$$

$$\theta = \tan^{-1}(0.05157)$$

$$\theta = 2.95°$$

When using your calculator with the trig functions, be sure your calculator is in "degrees mode." Some calculators and spreadsheets default to "radians mode."

If we consider that the surface of the earth is NOT flat, then the actual altitude of the receiving station will be lower than the value used, and the angle of elevation will be accordingly less. The resulting angle might even be negative, for example, when sending from a mountaintop to another distant mountaintop or a valley below.



Practice Exercises

Practice Exercises

Exercise 1

In electrical circuits with varying currents and voltages, the combined effect of resistance and reactance is called impedance. The impedance is related to the resistance and the reactance by the formula shown below.

$$Z^2 = R^2 + X^2$$

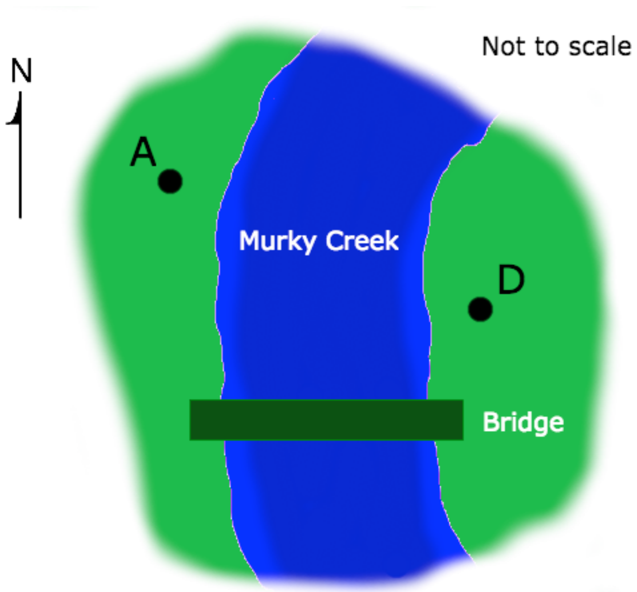
where Z is the impedance, in ohms,
R is the resistance, in ohms, and
X is the reactance, also in ohms.

a. The relationship above is equivalent to the Pythagorean formula that relates the sides of a right triangle. Draw and label a right triangle to represent the relation among the impedance, the resistance, and the reactance.

b. A loudspeaker is labeled as having an impedance of 8.0 ohms. Your measurement with an ohmmeter shows a resistance of 1.5 ohms. What is the reactance of the speaker?

Exercise 2

You need to string a cable across Murky Creek and need to determine the distance across. Suppose the two points can be labeled A and D, as shown in the sketch below. You find that, starting at point A, you can walk due south for 21 meters then turn due east and walk 22.5 meters across the bridge. Finally, walking due north 9 meters you reach the desired point D.



a. On your paper, make a sketch of the distances and directions traveled, similar to the sketch above. The final point should be labeled D.

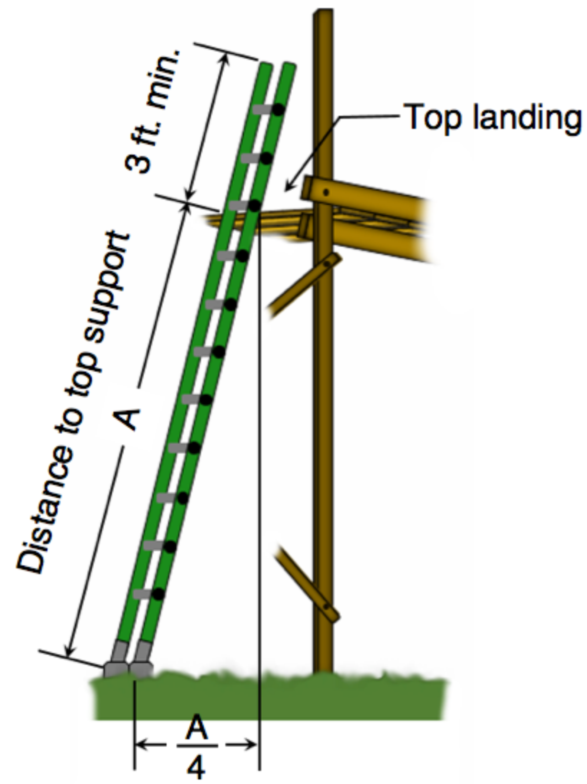
b. The segment joining point A to point D on your scale drawing is the distance you need to string cable. How many meters is it across Murky Creek, from point A to point D?

Exercise 3

A manual has the illustration shown here as a guideline for the safe use of an extension ladder. Suppose your ladder has a maximum extended length of 24 feet.

a. Using the guideline of having 3 feet extending above the top landing, what is the distance to the top support for your ladder?

b. Using the guideline shown in the drawing, what is the maximum landing height that you can safely reach with your ladder?



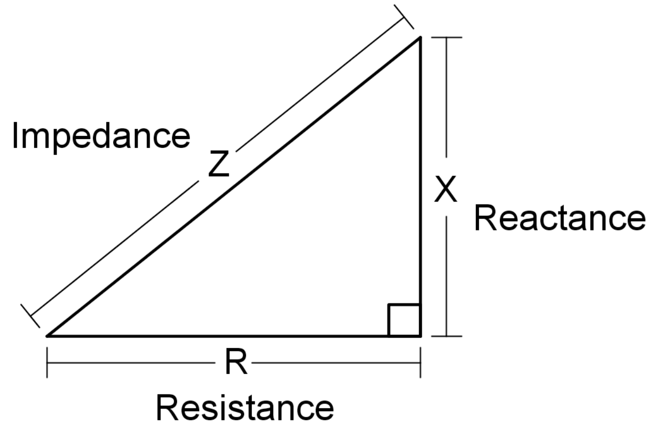
Exercise 4

You want to build a picture frame from a kit. Each of the four frame pieces is 12 inches long (on the longest edge). To check for “squareness” you plan to measure the diagonals. How long should the diagonal measurement be (at its longest point) for this frame?

Solutions to Practice Exercises

Solutions to Practice Exercises

1. a. The labels for the legs can be reversed, but the hypotenuse must correspond to the impedance Z.

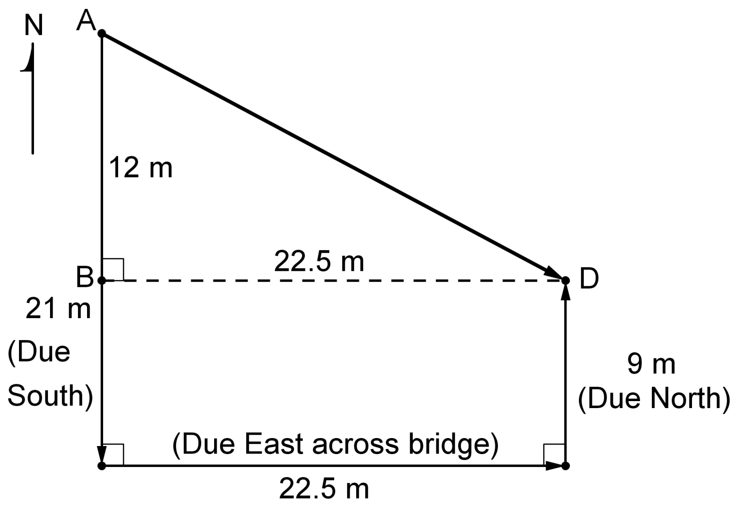


b. $Z^2 = R^2 + X^2$

$$x = \sqrt{(8.0 \text{ ohms})^2 - (1.5 \text{ ohms})^2}$$

$$x = 7.9 \text{ ohms (rounded)}$$

2. a.



b. Using the right triangle ABD shown in the drawing above:

$$AD = \sqrt{(AB)^2 + (BD)^2}$$

$$AD = \sqrt{(21 - 9)^2 + (22.5)^2}$$

$$AD = \sqrt{12^2 + (22.5)^2}$$

AD = 25.5 m

3. a. To have 3 feet above the top landing with a 24-foot ladder, you can have no more than 21 feet (that is, 24 ft - 3 ft) between the point at which the ladder touches the ground and the top support.

b. The drawing with the exercise shows a right triangle whose base is one-quarter the length of the hypotenuse. The hypotenuse was determined in part (a) to be 21 feet. Then use the Pythagorean formula to solve for the remaining leg.

$c^2 = a^2 + b^2$ (Solve for one leg, b)

$$b = \sqrt{c^2 - a^2}$$

Here $c = 21$ ft and $a = \frac{c}{4} = \frac{21}{4} = 5.25$ ft

$$b = \sqrt{(21)^2 - (5.25)^2}$$

b = 20.33 ft or 20' 4" (rounded)

This ladder's maximum safe landing height is 20' 4" from the ground.

4. Since the frame will be square, each pair of sides forms a right triangle. The diagonal will be the hypotenuse, so you can use the Pythagorean formula.

$$c^2 = a^2 + b^2$$

$$c = \sqrt{(12\text{ in})^2 + (12\text{ in})^2}$$

$c = \sqrt{288\text{ in}^2}$, or 17 in (rounded)

Or you may use the fact that the right triangle formed is a 45°–45° triangle. Then the hypotenuse is $\sqrt{2}$ times the length of one leg, or 12 $\sqrt{2}$ inches, yielding the same answer as above.

Career Video

Unlimited Learning Potential

A change in careers doesn't have to mean starting from scratch. Many college programs are designed to build on students' existing skills.

Special Graphs

11

Objectives

When you have completed this section, you should be able to do the following:

1. Read and interpret exponential graphs on linear and semilog scales
2. Graph exponential functions on semilog scales
3. Read and interpret polar graphs
4. Graph with polar coordinates
5. Convert back and forth from rectangular to polar coordinates

Scenario

Following recent flooding, measurements of coliform (a bacteria) levels in a waterway show slowly diminishing levels over the course of several days, as shown in the table here: counts of number of bacteria per 100 milliliters. You suspect an exponential decay process. Prepare a graph to verify your suspicions, and predict when the coliform count will drop below 40 per 100 ml.

Coliform Counts

t (days)	#/100ml
0	705
10	387
20	249
30	120
40	80

Before looking at the solutions, work through the lesson to further develop your skills in this area.

Graphs of Exponentials

Graphs of Exponentials

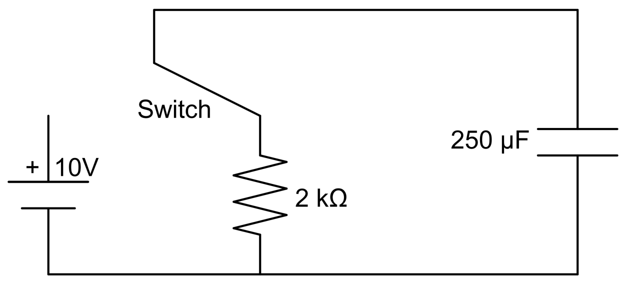
We have discussed exponential functions in previous sections, but now we will look at their graphical representations. The graphs of exponential functions differ from linear functions in that they do not have constant increases or decreases of *y* as *x* changes. In other words, there is not a constant slope, as is true with linear graphs. We will begin our study of exponential functions with a look at the graph of a common relationship in electrical circuit analysis: the discharging of a capacitor.

$$V = V_0 e^{-t/RC} \text{ or } \frac{V}{V_0} = e^{-t/RC}$$

where	V	=	Voltage across capacitor C after time (t)
	V_0	=	Battery voltage
	t	=	Time since charging process began
	R	=	Resistance in the series circuit
	C	=	Capacitance in the series circuit
		=	Ratio of capacitor voltage to battery voltage

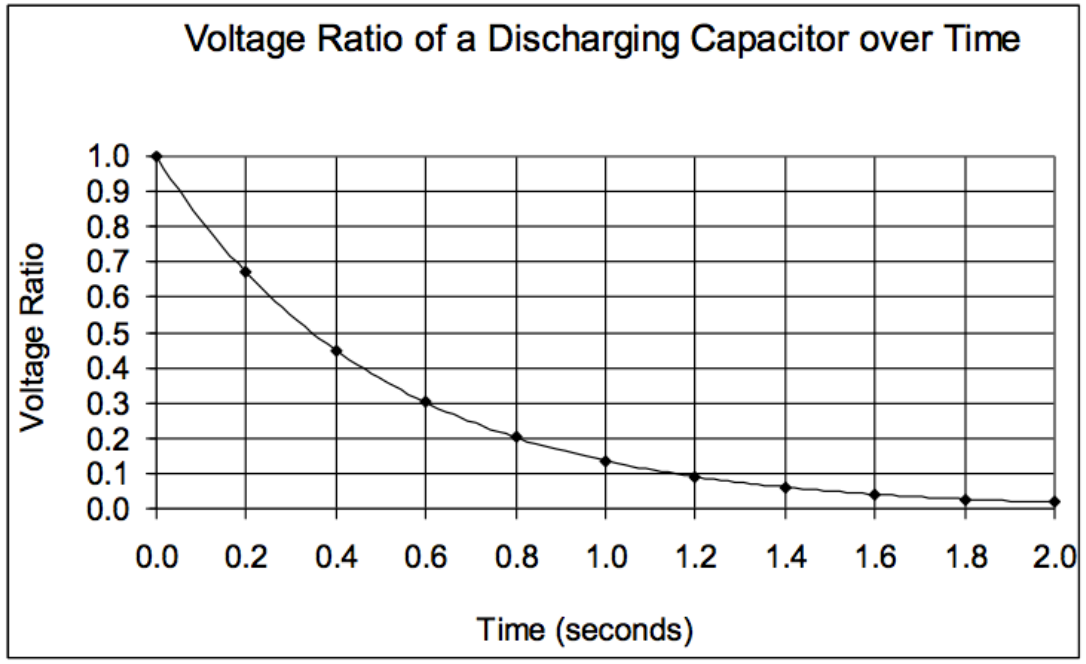
Example 1

A 10-volt (V_0 = 10 V) battery was used to charge a 250-μF capacitor. It is going to be discharged through a 2-k resistor. Below are a circuit diagram and a list of measured values of the voltage ratio (V/V_0) as the capacitor **discharges**.



Time (seconds)	Voltage Ratio
0.0	1
0.2	0.670
0.4	0.449
0.6	0.301
0.8	0.202
1.0	0.135
1.2	0.091
1.4	0.061
1.6	0.041
1.8	0.027
2.0	0.018

Here is a graph of this relationship as ordered pairs in the Cartesian coordinate system with Time on the x-axis and the Voltage Ratio on the y-axis.



Notice that the voltage ratio initially falls rapidly, but as time passes, the rate of decrease slows. It's almost as if it will never reach a value of zero. We say that it approaches zero asymptotically. A value called the time constant (τ) is very closely related to the shape of the curve and the rate at which the ratio decreases. The value of τ can be calculated by multiplying the resistance (in ohms) by the capacitance (in Farads): RC. Using τ, we can rewrite the equation stated earlier:

$$\frac{V}{V_0} = e^{-t/\tau}$$

This simplified version deals only with our variables (time and voltage ratio) and the time constant (τ). You will learn more about the properties of the time constant and the uses of this equation in another course. For now you can treat the time constant as an important part of exponential functions, just as you know the slope is an important part of linear functions.

Example 2

The intensity level of sound is modeled by a logarithmic function.

$$IL = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where IL = Intensity level (dB)

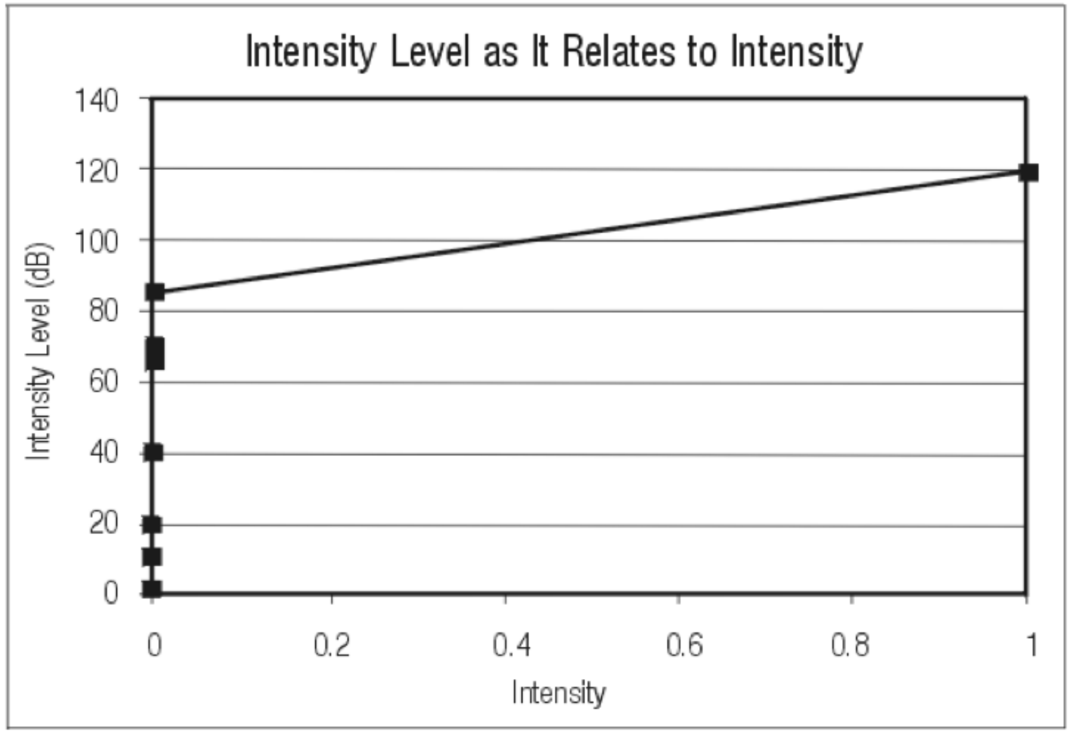
I = Intensity (W/m^2) of a particular sound

I_0 = Standard threshold of audibility (hearing) = 1 10–12 (W/m^2)

The standard threshold of audibility (I_0) is a constant value measured at 10–12 W/m^2. For given descriptive sound levels, the following Intensity and Intensity Levels were measured as seen in the table below:

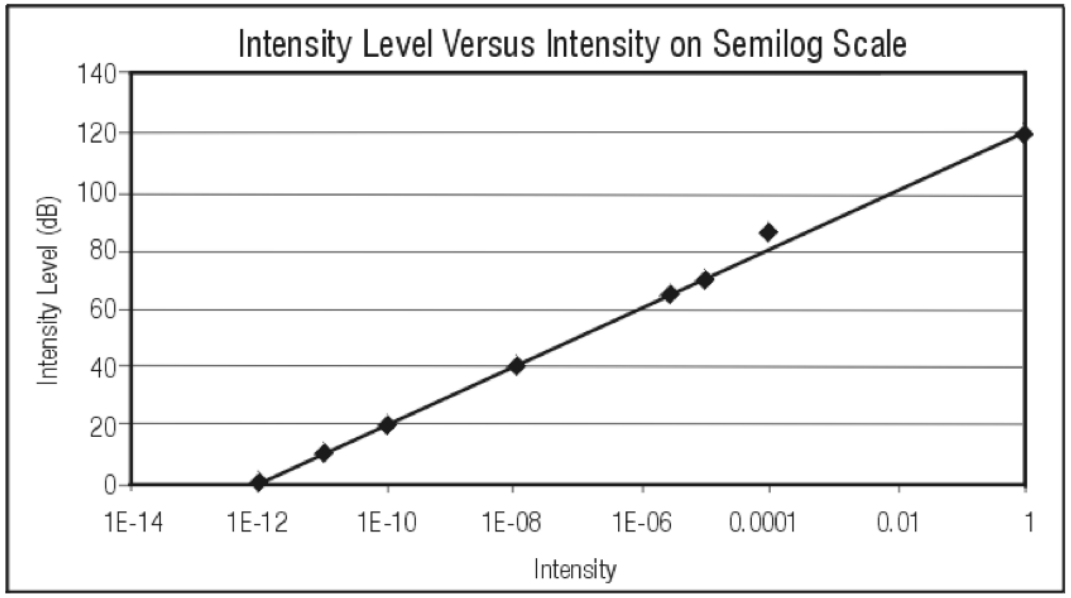
Sound	Intensity (W/m^2)	Intensity Level (dB)
Painful	1	120
Train	1×10^{-4} (or 1E-04)	85
Busy street traffic	1×10^{-5} (or 1E-05)	70
Typical conversation	3×10^{-6} (or 3E-06)	65
Quiet radio	1×10^{-8} (or 1E-08)	40
Whisper	1×10^{-10} (or 1E-10)	20
Rustle of leaves	1×10^{-11} (or 1E-11)	10
Hearing threshold	1×10^{-12} (or 1E-12)	0

From these values, we can attempt a graph with linear scales along the x- and y-axes. But, as the plot shows below, this does not come without difficulties.



The problem with this graph is that many of the x-values are so small that they all appear to be zero. If you were to drop the point (1, 120) you would have the same difficulty with the largest point being visible and all the others apparently on the y-axis. Although this graph is a function, it does not appear to be one. So in this case the graph is not very useful.

There is an alternative though. We can represent the data on a semilog plot. This means one axis (x or y) is “logged.” For the data above, if we use the log of the x-values, or simply show the x-axis on a logarithmic scale, we get a much different graph, as seen below.



This new plot results in a straight line! What is most useful about this graph is that the points are distinguishable. However, it may be difficult to see how the “logged” axis is working. Below is a table of arbitrary (yet simple) data and the accompanying graphs on a regular (both axes linear) scale and a semilog (one axis linear and one logarithmic) scale.

Example 3

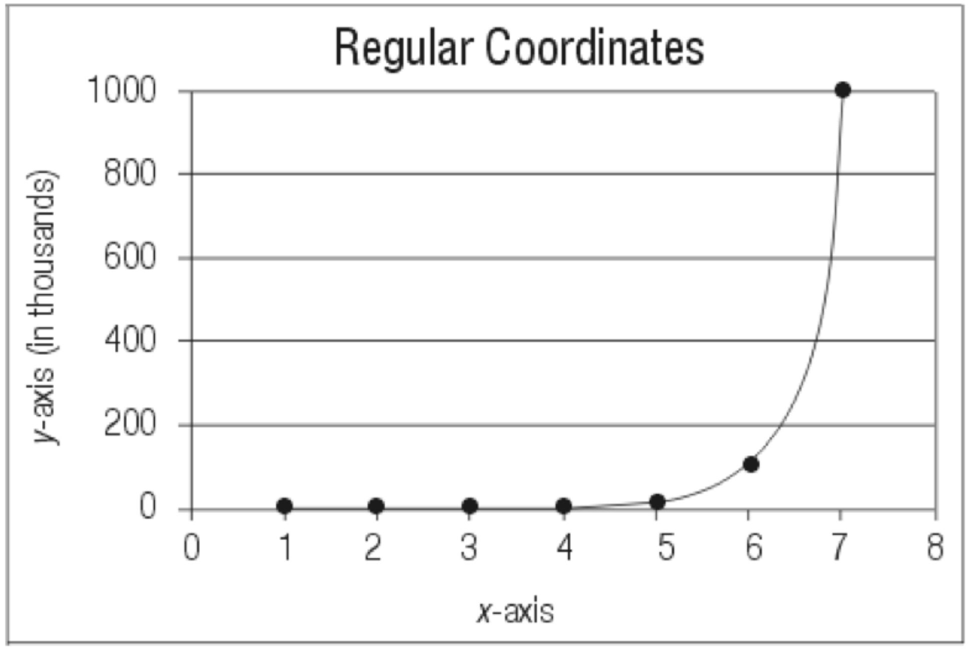
Plot the following table of data with:

a. 2 linear scales.

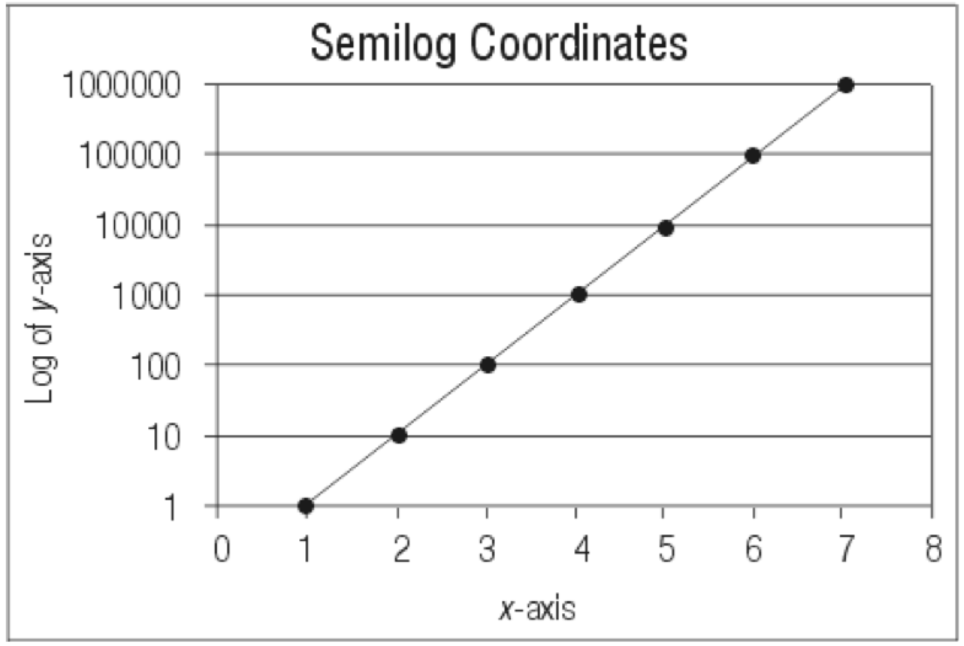
b. The x-axis as linear and the y-axis as logarithmic (semilog).

x	*y*
1	1
2	10
3	100
4	1000
5	10000
6	100000
7	1000000

a.



b.



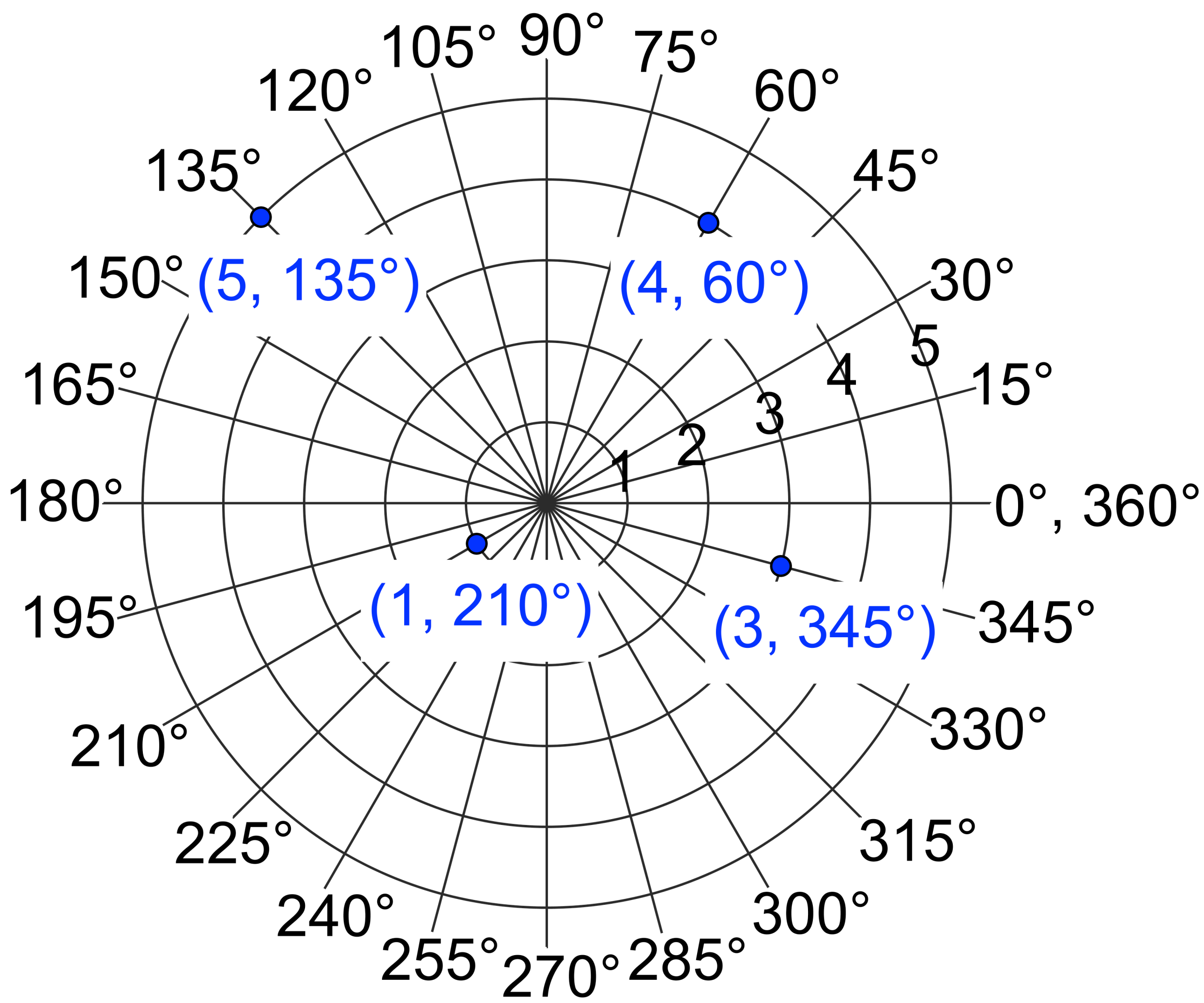
This is another situation in which the semilog plot is better because the information is much easier to read. Can you see how the y-axis covers a very large range of values by basically representing on the graph just the exponent of ten for the values, that is the log y values?

Think about this: What span of values is covered by the first two increments on the y-axis: from 1 to 10? Then, what span of values is covered by the last two increments on the y-axis: from 100,000 to 1,000,000?

Polar Coordinates

Polar Coordinates

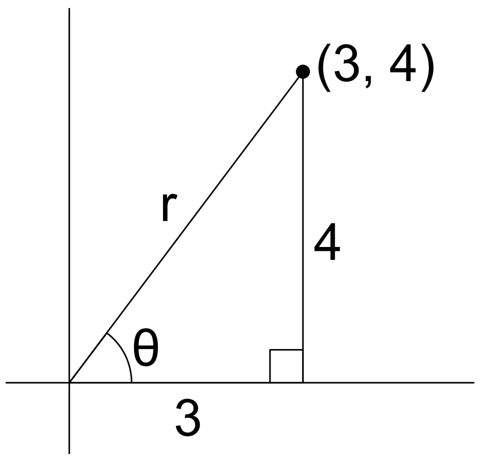
Another important coordinate system is polar coordinates. It has many important uses. The basis for this system is that points are plotted, not in (x, y), but in (r, θ). The first coordinate, r, is the radius and measures the distance from the origin to the point being plotted. The second, θ, is the angle measured counterclockwise from the positive x-axis (as you practiced in the angle measurement section of this book). The figure below shows some points plotted in polar coordinates.



Conversion

Rectangular to Polar

Oftentimes, we need to convert from rectangular coordinates to polar coordinates and vice versa. The process of converting the Cartesian point (3, 4) to polar coordinates is demonstrated below.



$r^2 = 3^2 + 4^2$ (Pythagorean theorem)

$r = 5$

$\tan\theta = \frac{4}{3} \qquad \left(\tan\theta = \frac{\text{opposite}}{\text{adjacent}}\right)$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\theta = 53.1°$ (rounded)

Therefore, (3, 4) in rectangular coordinates is equivalent to (5, 53.1°) in polar coordinates.

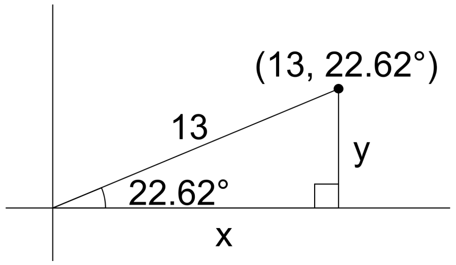
$(3, 4) \Leftrightarrow (5, 53.1°)$

In general, we can now make the following assumptions:

$r = \sqrt{x^2 + y^2}$ AND $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Polar to Rectangular

Converting from polar to rectangular coordinates is another important skill. Here we will convert the polar point (13, 22.62°) to Cartesian coordinates.



$$\cos 22.62° = \frac{x}{13} \quad \left(\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}\right)$$

$$x = 13\cos 22.62°$$

$$x = 12\ (\text{rounded})$$

$$\sin 22.62° = \frac{y}{13} \quad \left(\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}\right)$$

$$y = 13\sin 22.62°$$

$$y = 5\ (\text{rounded})$$

Therefore, (13, 22.62°) in polar coordinates is equivalent to (12, 5) in rectangular coordinates.

$$(13, 22.62°) \Leftrightarrow (12, 5)$$

In general, we can now make the following assumptions:

$$x = r\cos\theta \qquad \text{AND} \qquad x = r\sin\theta$$

Example 4

Given: $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x = r\cos\theta$, AND $y = r\sin\theta$

a. Convert $(7\sqrt{2}, 315°)$

b. Convert (3, 5)

Solution

a. r = 7 and = 315°

$x = r\cos\theta$ | $y = r\sin\theta$

$x = (7\sqrt{2})\cos(315°)$ | $y = (7\sqrt{2})\sin(315°)$

$x = 7$ | $y = -7$

$(x, y) = (7, -7)$

b. x = 3 and y = 5

$r = \sqrt{x^2 + y^2}$ | $\theta = \tan^{-1}\frac{y}{x}$

$r = \sqrt{3^2 + 5^2}$ | $\theta = \tan^{-1}\left(\frac{5}{3}\right)$

$r = \sqrt{34}$ | $\theta = 59°$

$r = 5.8$

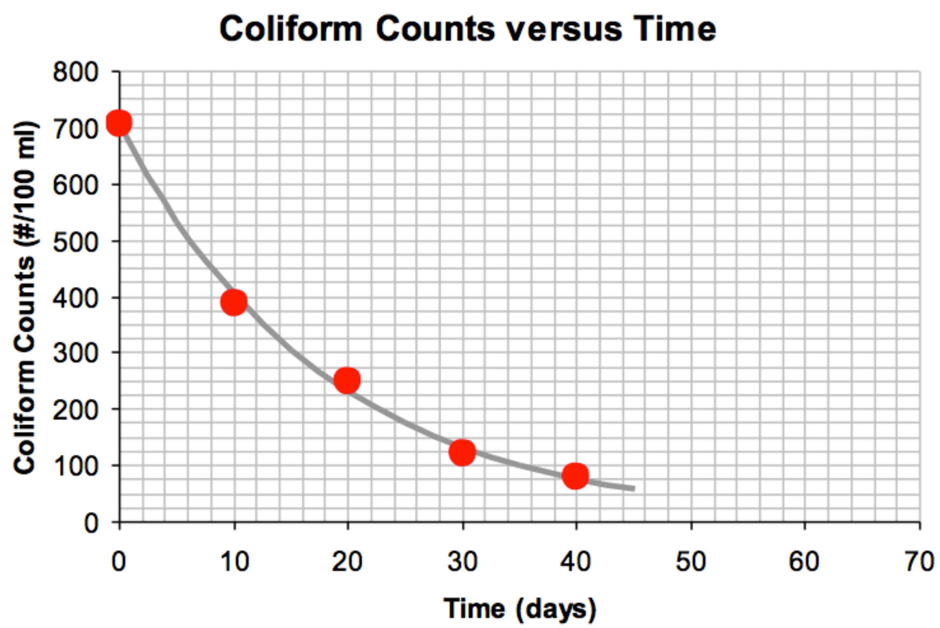
$(r, \theta) = (5.8, 59°)$

Solution to Scenario Question

Solution to Scenario Question

A graph of the data clearly shows a non-linear trend to the coliform counts over time. This supports your theory that the data may be exponential in nature. A better test of your theory is a logarithmic graph of the data.

Also, while we can probably draw a smooth curve through the plotted values (as shown), it would be difficult to predict when the counts might drop to a specific threshold, like 20 per 100 ml. Again, as we'll see, a logarithmic graph of the data should be explored.

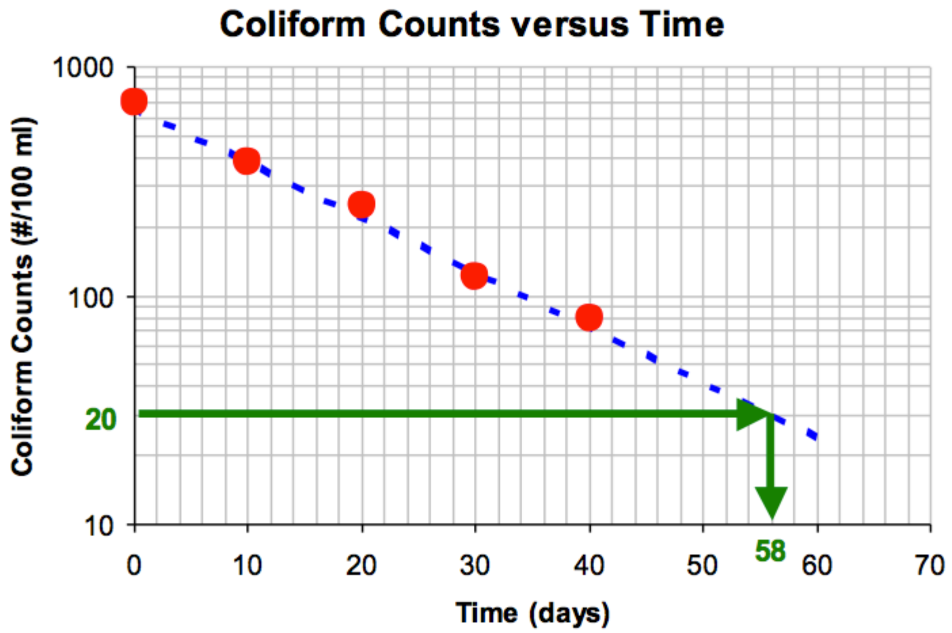


A Plot on Normal Graph Paper

Since we suspect that the *coliform counts* are exponentially decaying, but *not the time values*, we use **semi**-log paper, where the y-axis is scaled logarithmically but the x-axis is scaled arithmetically (or linearly).

Looking at the result on semi-log graph paper, as shown here, the trend of the data appears almost linear—a sign that there is likely an exponential relationship between *Counts* and *Time*.

Also, the linear nature of the trend when plotted on semi-log paper makes it possible to more reliably predict the trend of the data, as shown by the line sketched in the graph here.



A Plot on Semi-Log Graph Paper

Thus, we could say that the counts would probably reach 20 (per 100 ml) near day 58, as depicted in the graph here. Such a prediction would be much harder to make using a normal (non-logarithmic) graph of the data.

Practice Exercise

Practice Exercises

Exercise 1

In the first example of this section the exponential equation was graphed in Cartesian coordinates. Graph the same data in semilog, with the x-axis (time) on a linear scale and the y-axis (voltage ratio) on a logarithmic scale.

Exercise 2

A sound intensity level in decibels (dB) can also be expressed as a logarithmic function of an intensity ratio. The relation is shown below:

$$IL = 10\log\left(\frac{I}{I_o}\right)$$

where IL = Intensity level (dB)

$\frac{I}{I_0}$ = Sound intensity ratio of actual sound intensity to a reference sound

a. Complete the table below using the formula above:

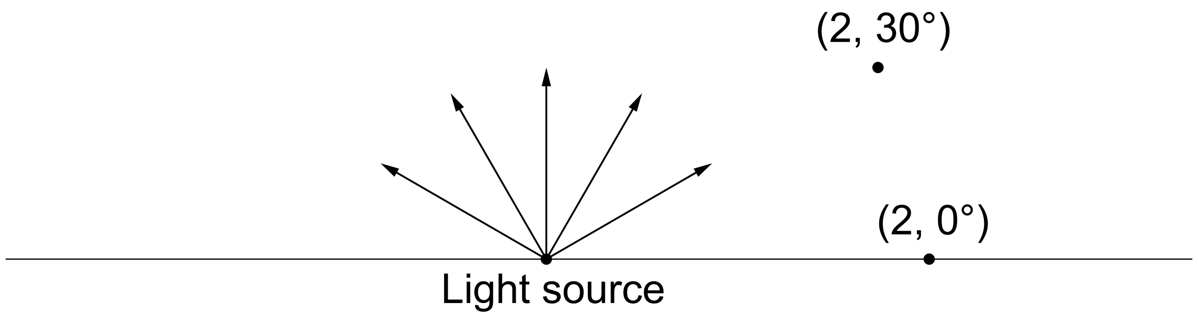
$\frac{I}{I_o}$ Ratio	1	10	20	30	40	50	60	70	80	90	100
Intensity Level (dB)											

b. Graph the above data on a regular scale (two linear axes) with the voltage ratio on the x-axis and the intensity level on the y-axis.

c. Graph again on a semilog scale (with the x-axis on a logarithmic scale and the y-axis on a linear scale).

Exercise 3

The area around a point source of light is measured for light intensity as seen in the figure below (points are plotted in polar coordinates (r, θ)).



Fraction of Original Light Intensity (I_o) at Various Values of the Radius (*r*) and Degree (θ)

r \ θ	0°	30°	60°	90°	120°	150°	180°
2 cm	0.24	0.25	0.25	0.26	0.25	0.24	0.24
4 cm	0.05	0.06	0.06	0.07	0.06	0.05	0.05
6 cm	0.02	0.03	0.03	0.03	0.03	0.03	0.02
8 cm	0.01	0.02	0.02	0.02	0.02	0.02	0.01
10 cm	0.01	0.01	0.01	0.01	0.01	0.01	0.01

An instrument that measures light intensity is placed 2 cm, 4 cm, 6 cm, 8 cm, and 10 cm away. At each distance it is placed at 30° increments from 0° to 180° as demonstrated above. The table below shows the results of these measurements.

a. Graph the points where light intensity was measured at 2 cm, 6 cm, and 10 cm (at all degrees listed) on a polar coordinate system.

b. What do you notice about the intensities as they relate to the radii and degree measurements?

Exercise 4

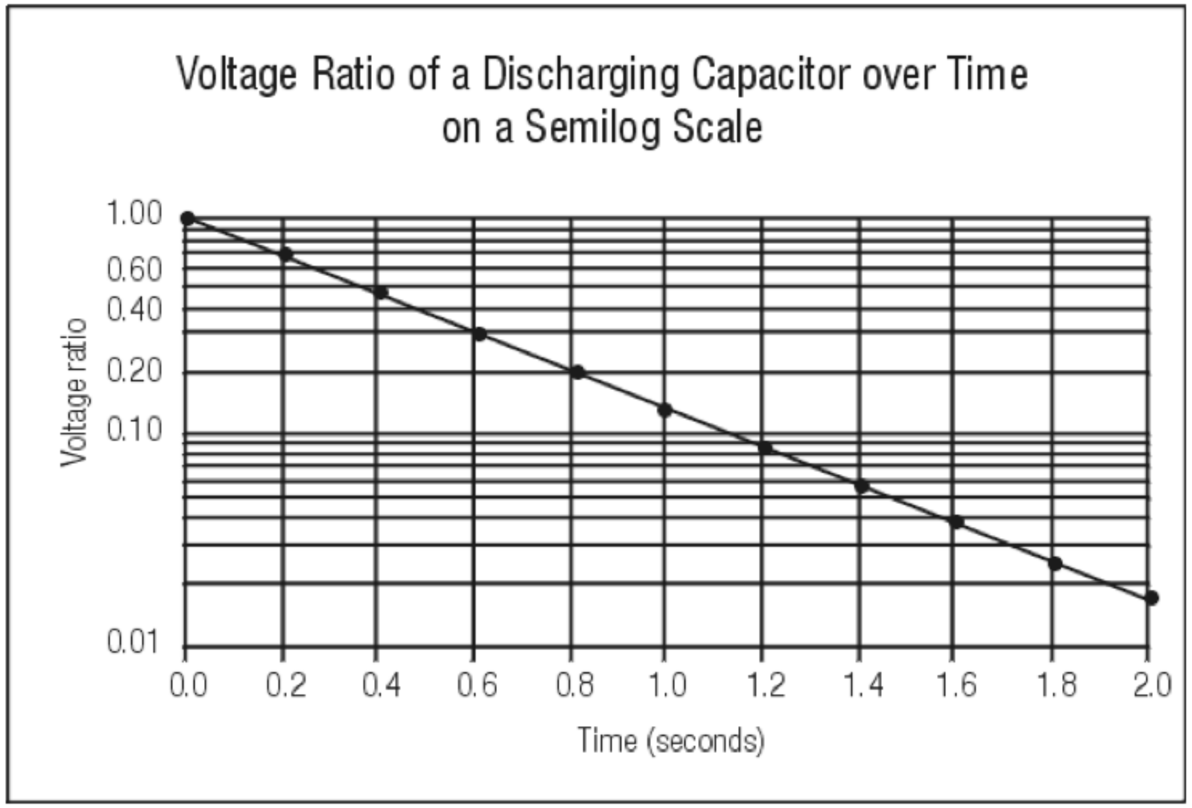
a. Convert (17, 81°) to rectangular coordinates.

b. Convert (32, 9) to polar coordinates.

Solutions to Practice Exercises

Solutions to Practice Exercises

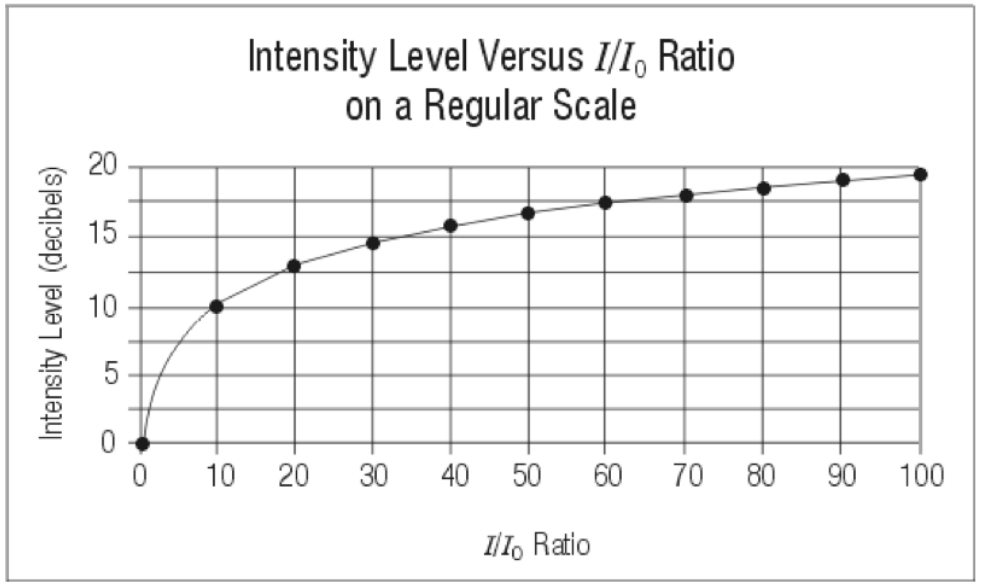
1.



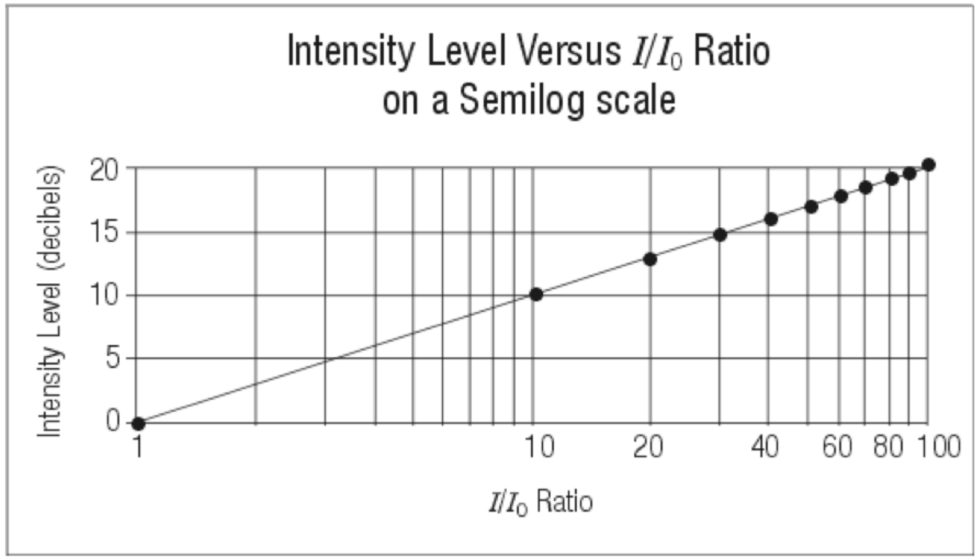
2. a.

$\frac{I}{I_0}$ Ratio	1	10	20	30	40	50	60	70	80	90	100
Intensity Level (dB)	0	10	13.0	14.8	16.0	17.0	17.8	18.5	19.0	19.5	20.0

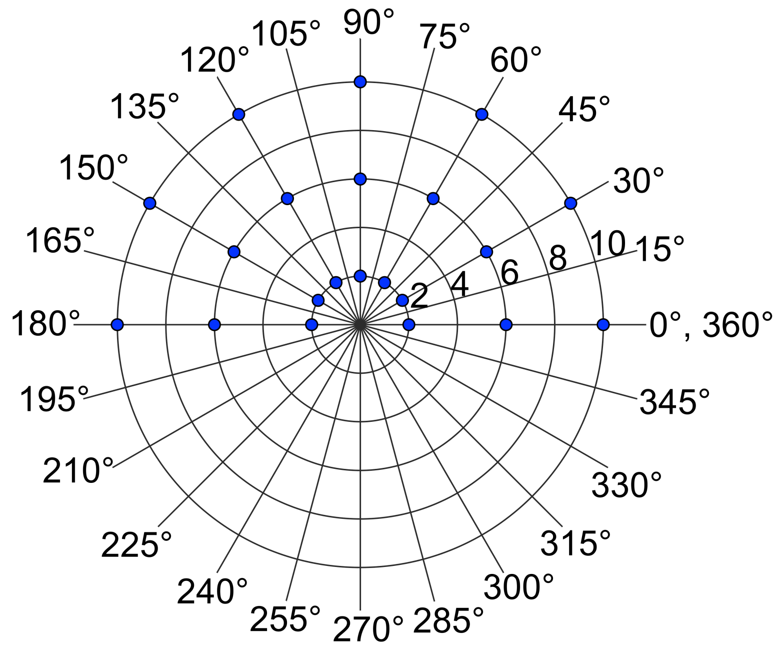
b.


Intensity Level Versus I/I_0 Ratio
on a Regular Scale
Intensity Level (decibels)
20
15
10
5
0
0 10 20 30 40 50 60 70 80 90 100
I/I_0 Ratio

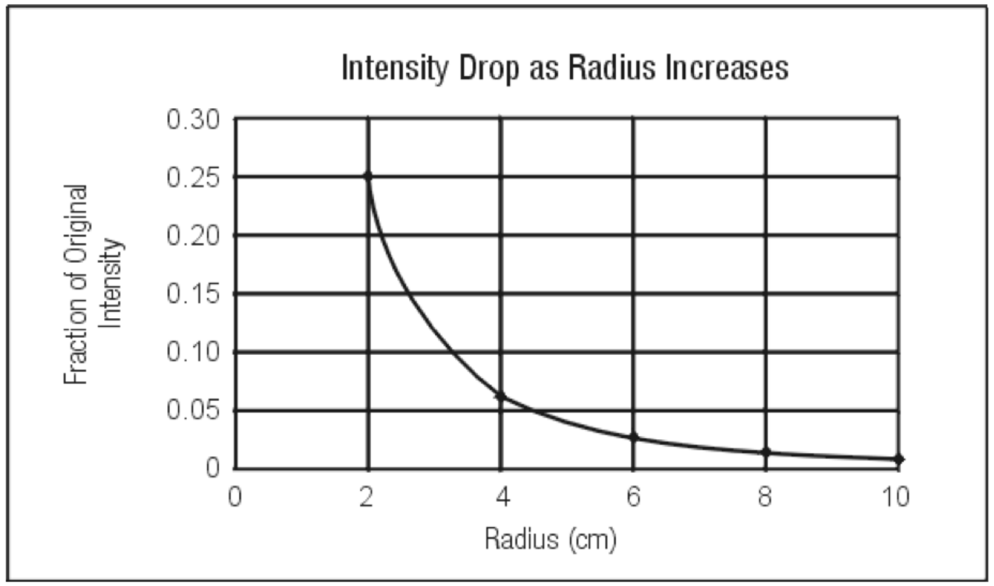
c.


Intensity Level Versus I/I_0 Ratio
on a Semilog scale
Intensity Level (decibels)
20
15
10
5
0
1 10 20 40 60 80 100
I/I_0 Ratio

3. a.


90°
105°
75°
120°
60°
135°
45°
150°
30°
165°
15°
180°
0°, 360°
195°
345°
210°
330°
225°
315°
240°
300°
255°
270°
285°
2
4
6
8
10

b. The intensity decreases as the radius increases (they are inversely proportional) and it is essentially independent of changes in the angle (θ). A more significant graph would use (r, I), where I is the fraction of the original intensity (I_0) remaining.



In this graph, it is clear that as the radius increases the light intensity decreases. In fact, we can now determine a relation between the radius (r) and the percent of original intensity ($\frac{I}{I_0}$) as follows:

$$I = \frac{I_0}{r^2}$$

4. Given: $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x = r\cos\theta$, AND $y = r\sin\theta$

a. $(r, \theta) = (17, 81°)$

$r = 17$

$\theta = 81°$

$x = (17)\cos(81°)$ $y = (17)\sin(81°)$

$x = 2.66$ $y = 16.79$

$(x, y) = (2.66, 16.79)$

b. $(x, y) = (32, 9)$

$x = 32$

$y = 9$

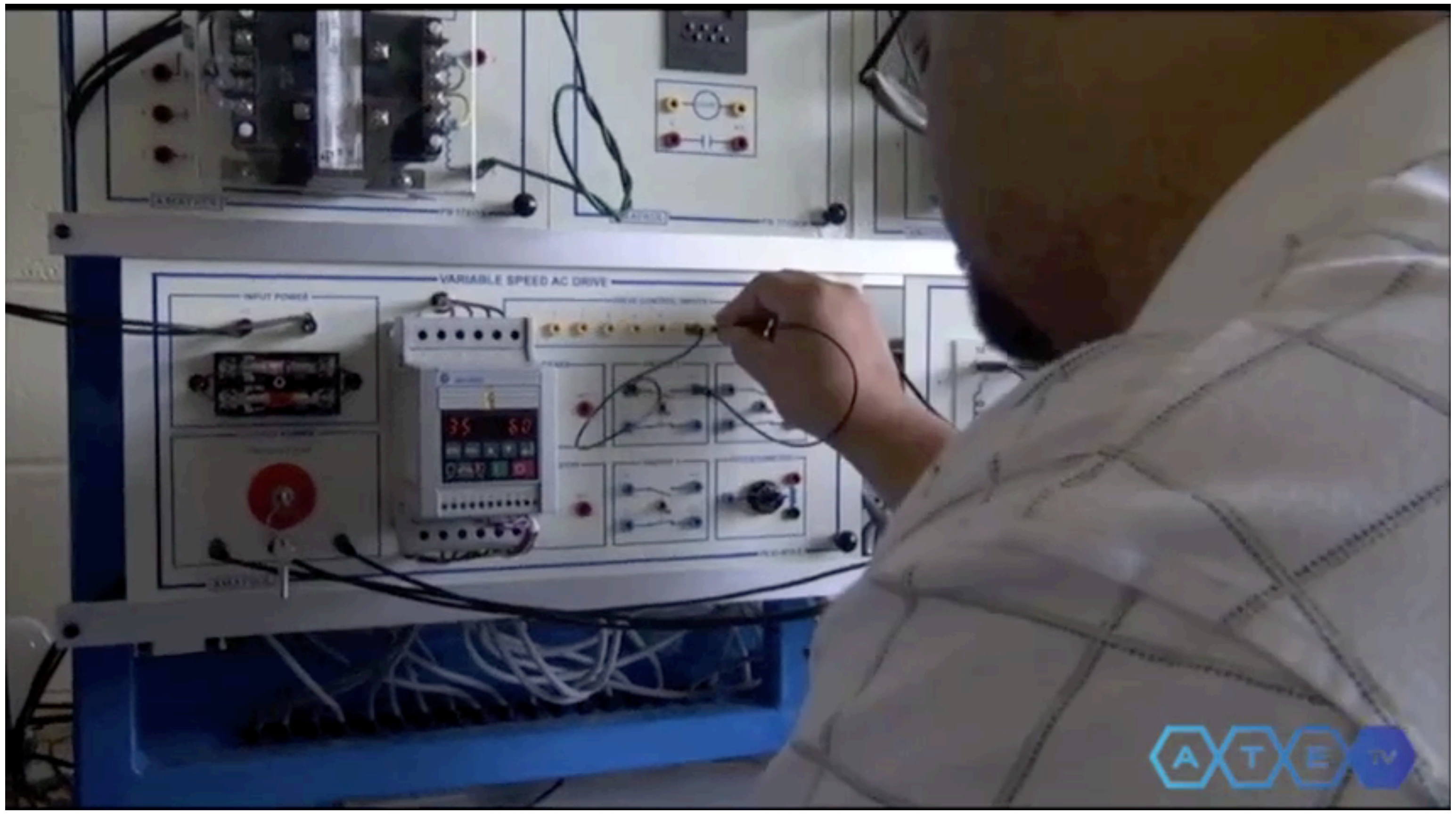
$r = \sqrt{32^2 + 9^2}$ $\theta = \tan^{-1}\left(\frac{9}{32}\right)$

$r = 33.24$ $\theta = 15.71°$

$(r, \theta) = (33.24, 15.71°)$

Career Video

Industrial Maintenance Class Produces Multi-Skilled Technicians



Visit a class where students are mastering everything from welding and HVAC skills to robotics and electronics. It's sometimes known as "industrial maintenance on steroids."

Sinusoidal Motion

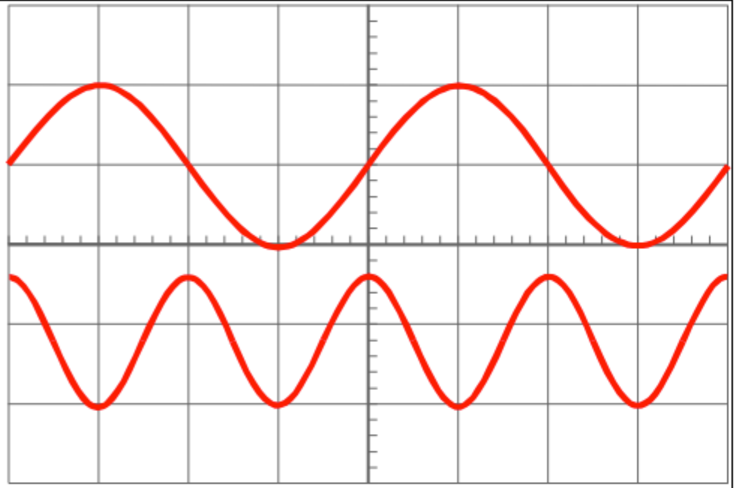
12

Objectives

When you have completed this section, you should be able to do the following:

1. Calculate the frequency and period of a periodic function depicted by a graph.
2. Calculate the speed, wavelength, and amplitude of a traveling wave depicted by a graph.

Scenario



A pair of electronic probes samples the voltage signals at two test points in a circuit. The resulting traces are displayed on a dual-trace oscilloscope screen, as shown here. The oscilloscope settings are:

- Horizontal scale: 2 msec/div
- Vertical scale: 0.1 volts/div

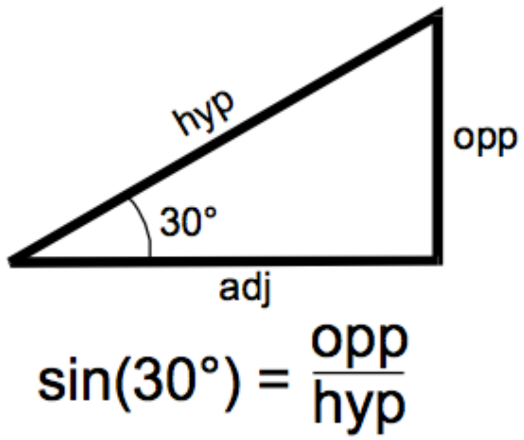
What are the amplitude, frequency, and period of the signals?

Before looking at the solution, work through the lesson to further develop your skills in this area.

Periodic Motion

Periodic Motion

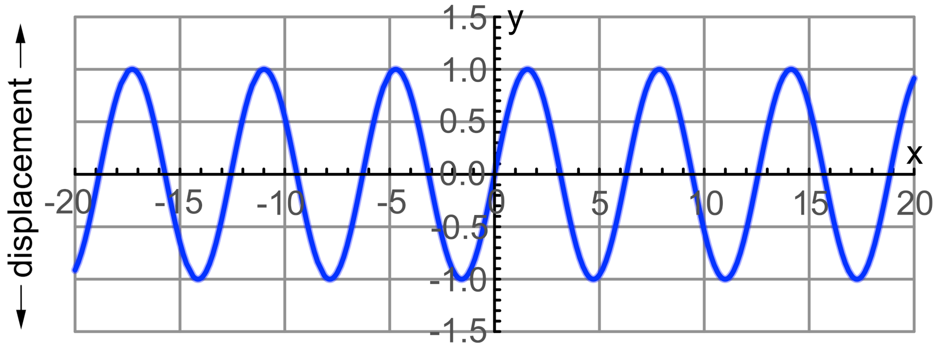
You've seen how the sine and cosine functions describe the ratios of sides of a right triangle and the angle between them. For example, sin(30°) = 0.5 is the ratio between the side of a right triangle opposite a 30° angle and its hypotenuse.



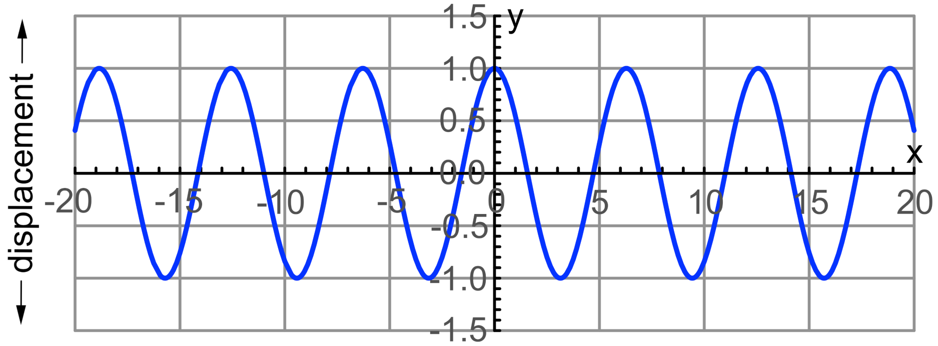
The argument for the sine and cosine functions can be either in degrees or radians. Thus, $\sin(30°) = \sin\left(\frac{\pi}{6}\right) = \sin(0.5236) = 0.5$.

Note: When you calculate sine and cosine values with your calculator, be careful to know whether your calculator is in "degrees" or "radians" mode.

Any motion that repeats itself in equal intervals of time is called periodic motion. Examples that come to mind are pendulums, piano strings, rolling wheels, and so forth. Engineers use the sine and cosine functions to describe periodic motions like these. Graphs of y = sin(x) and y = cos(x) are shown below (x-values are in radians).



Graph of y = sin(x)



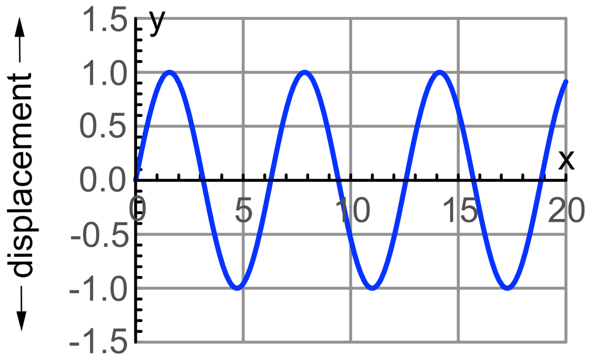
Graph of y = cos(x)

Notice some important facts about these two functions.

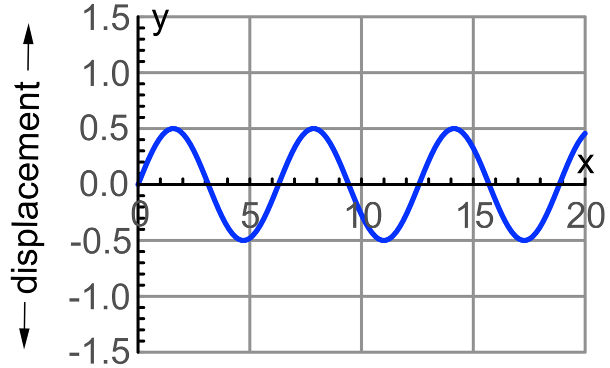
- They represent a *displacement* above or below a central or neutral position or value.
- They clearly “repeat,” for both smaller and larger values of x.
- They repeat in a regular way, about every 6.3 units (or 2π radians) of x.
- Displacements vary smoothly between values of –1 and +1, with an average of 0.
- The only difference between the two curves is a slight horizontal “shift”. Actually, if you shifted the cosine curve $\frac{\pi}{2}$ units to the right, it would overlap the sine curve. Mathematically, we say that $\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$.

Amplitude

Compare the graphs of the two functions below



Graph of y = sin(x)



Graph of y = 0.5sin(x)

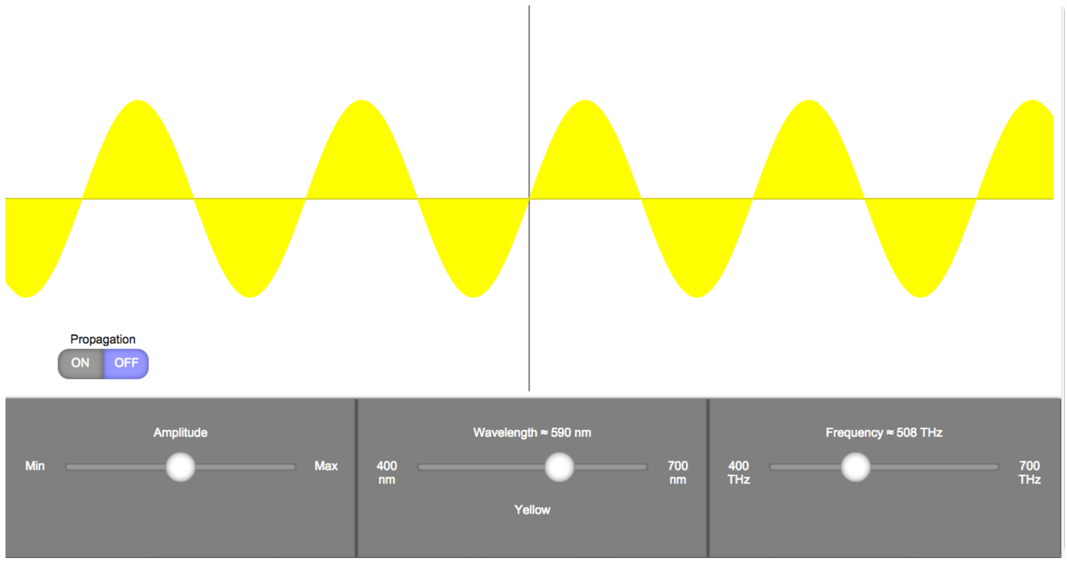
The right graph is shaped essentially the same as the left except for different maximum and minimum values of displacement. For simple periodic functions like sin(x) and cos(x) where the average value is zero, we call this maximum displacement value the amplitude. Typically, the amplitude will be a value that multiplies the sine (or cosine) term in a periodic function.

y = (1)sin(x)   y = (0.5)sin(x)

Amplitude is 1.   Amplitude is 0.5.

Note that the sine wave varies from a positive value (1 Amplitude) to a negative value (–1 Amplitude). So, the overall height of the curves is actually *twice the amplitude*.

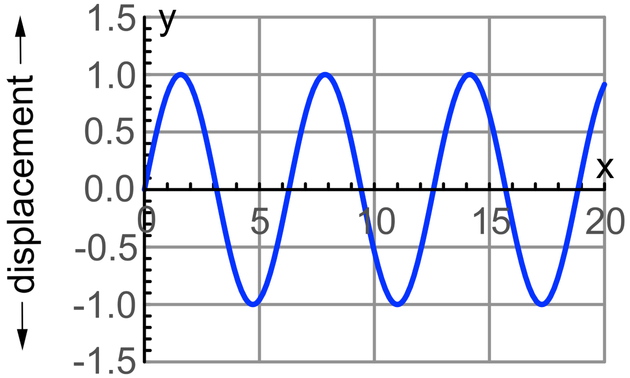
INTERACTIVE 12.1 Properties of a Sine Wave



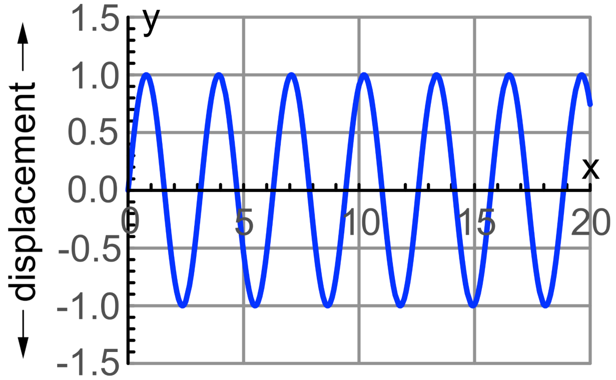
Click or Tap to Interact

Frequency

Compare the graphs of the two functions below.



Graph of y = sin(x)



Graph of y = sin(2x)

Again, the graphs are very similar, but the right one has more “cycles” over the same span of x-values. **The *frequency* of a periodic function is a measure of the number of cycles per range of x-values.**

Notice that the second graph depicts a function that yields twice as many cycles over the same span of x-values as the first graph. In general a multiplier on the x-term has the effect of changing the frequency of the periodic function.

y = sin(1x)   y = sin(2x)

Frequency is 1 cycle per 2 radians.   Frequency is 2 cycles per 2 radians.

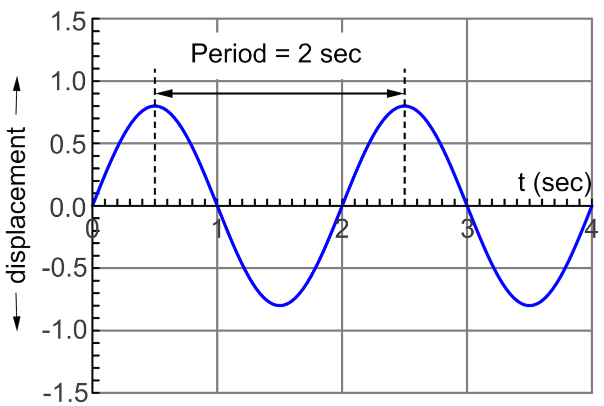
Period

Periodic functions are often based on time, t (typically measured in seconds). In that case, we can define the periodic function in a more useful way.

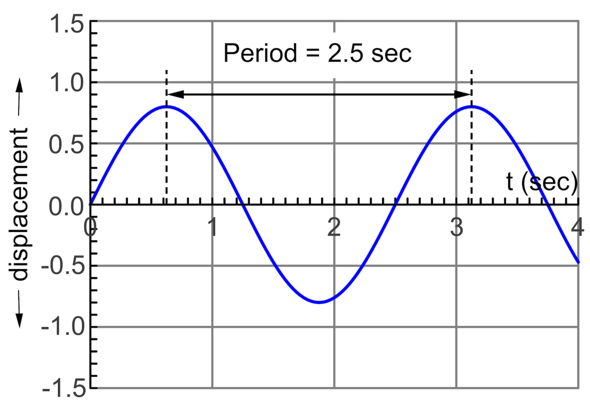
$$y = A \cdot \sin(2\pi f \cdot t)$$

where y = periodic displacement based on time, t measured in seconds
A = amplitude
f = frequency, in cycles per second

And, we can glean a bit more from the graph of the function.



Graph of $y = 0.8 \sin(2\pi(0.5)t)$



Graph of $y = 0.8 \sin(2\pi(0.4)t)$

In the first graph above, we see that the frequency is 0.5 cycles per second. In other words, after one second has passed, we expect the periodic function to have completed half of a cycle. So, after 2 seconds it will complete one full cycle. The period is the time required to complete one full cycle. Thus, the period in the first graph above is 2 seconds.

There is a simple and very useful relationship that connects the period T and the frequency f:

$$T = \frac{1}{f}$$

So, we can also write periodic functions based on time t in two ways: using either the frequency or the period.

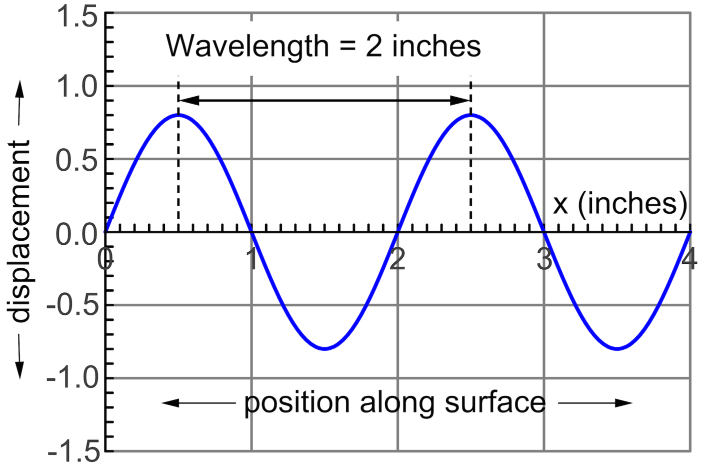
$$y = A \sin(2\pi f \cdot t) \quad \text{or} \quad y = A \sin\left(\frac{2\pi}{T} \cdot t\right)$$

where y = periodic displacement based on time, t (in seconds)
A = amplitude
f = frequency, in cycles per second
T = period, in seconds

Can you identify the frequency and period in the second graph?

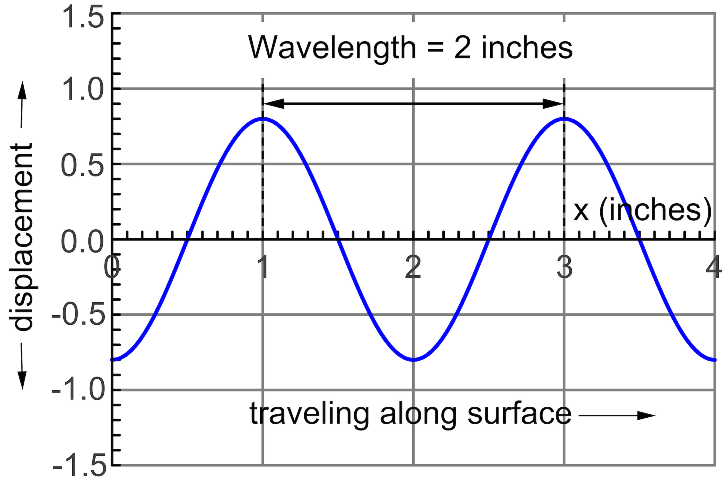
Wavelength

Often we encounter periodic functions that describe waves that travel through a medium, like air or water. For example, consider the ripples moving across the calm surface of a pond. A side-view snapshot of the surface might appear as shown below.



We can measure the *wavelength* of this traveling wave as the distance between peaks (or the distance between troughs). The symbol usually used for wavelength is the Greek letter lambda (λ).

Now, suppose we take another snapshot 0.1 second later. Since this is a traveling wave, the peaks have moved, as shown below.



Comparing the two graphs, we see that the peaks (and troughs) have all moved 0.5 inch in 0.1 second. We can calculate the wave speed, c.

$$c = \frac{\Delta x}{\Delta t} = \frac{0.5 \text{ in}}{0.1 \text{ sec}} = 5 \text{ in per sec}$$

There is a very useful relationship that connects wavelength, wave speed, and frequency:

$$c = \lambda f$$

So, for our example, we can solve for f and calculate the frequency of the pond ripples.

$$f = \frac{c}{\lambda}$$

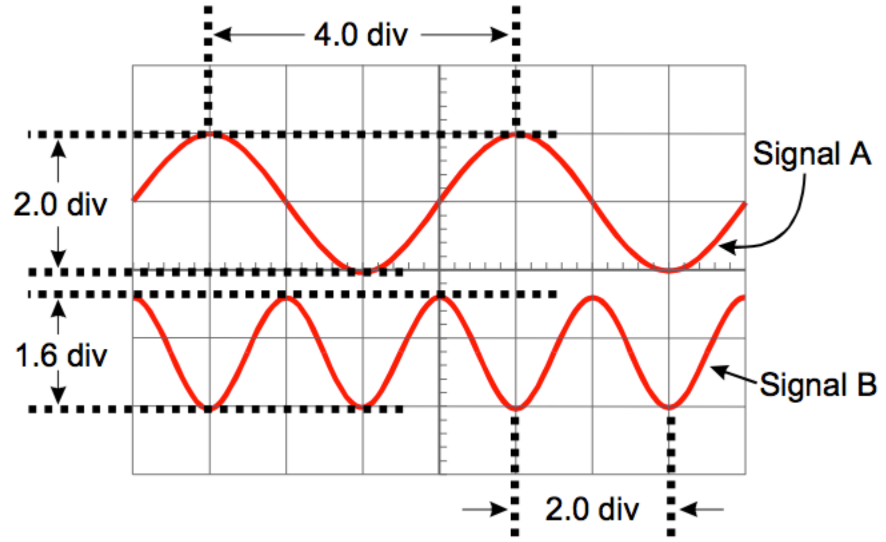
$$f = \frac{5 \frac{\cancel{in}}{sec}}{2 \cancel{in}}$$

$$f = 2.5 \frac{cycles}{sec}$$

Solution to Scenario Question

Solution to Scenario Question

An oscilloscope provides a snapshot of an electrical wave. The horizontal scale depicts the wave changing over time. The vertical scale depicts the voltage of the wave. It is common to have two (or more) "traces" on the screen at one time. That's why it's called a "dual-trace" oscilloscope.



Amplitude: We recognize the vertical "size" of the signals as the amplitude. For Signal A, we see that the overall height of the signal is 2.0 divisions. Recall that the overall height is actually twice the amplitude. So, the amplitude of Signal A is actually $\frac{2.0\,\text{div}}{2} = 1.0\,\text{div}$. Since we're given that the vertical scale is 0.1 volts/div, we know the amplitude of Signal A is 0.1 volts.

Similarly, from the display we see that Signal B is 1.6 divisions "peak-to-trough." So, the amplitude of Signal B is $\frac{1.6\,\cancel{\text{div}}}{2} \times 0.1\frac{\text{volts}}{\cancel{\text{div}}} = 0.080\,\text{volts}$.

Period: We recognize the period as the length of time between the peaks (or the troughs, or any other corresponding part of the repeating wave form). For Signal A, we see the wave repeating after a length of time corresponding to 4.0 divisions. So, the period of Signal A is $4.0\,\cancel{\text{div}} \times 2.0\frac{\text{msec}}{\cancel{\text{div}}} = 8.0\,\text{msec}$.

Similarly, from the display we see that Signal B is 2.0 divisions, "trough-to-trough." So, the period of Signal B is $T_B = 2.0\,\cancel{div} \times 2.0\frac{msec}{\cancel{div}} = 4.0\,msec.$

Frequency: We can't read the frequency directly from the oscilloscope display. However, knowing the period of the signal we can calculate the frequency! So, for Signal A:

$$f_A = \frac{1}{T_A} = \frac{1}{8.0\,\cancel{msec}} \times \frac{1000\,\cancel{msec}}{1\ sec} = 125 \text{ cycles per second}$$

Note that we've converted milliseconds to seconds. Similarly, then, for Signal B,

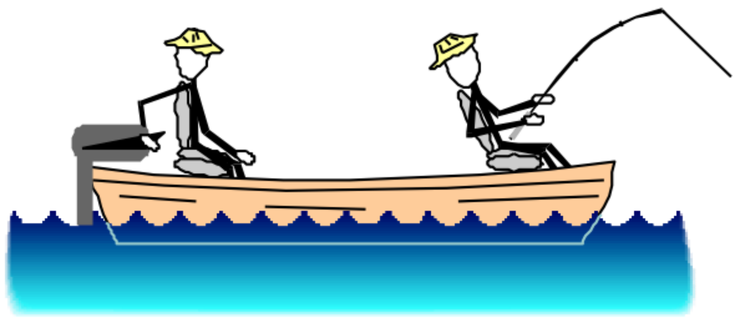
$$f_B = \frac{1}{T_B} = \frac{1}{4.0\,\cancel{msec}} \times \frac{1000\,\cancel{msec}}{1\ sec} = 250 \text{ cycles per second}$$

Practice Exercises

Practice Exercises

Exercise 1

Water waves pass by a 16-foot fishing boat. A fisherman notices that there are exactly 13 crests from front to back of the boat, and that the vertical distance between a crest and trough is about one-fourth as high as the boat. The boat is 18 inches high. Find the wave amplitude and wavelength of the water waves.



Exercise 2

The fisherman from Exercise 1 looks at his watch and times those waves passing by. (Yes, he's obviously having a slow day!) He counts 6 crests passing by in 10 seconds. Calculate the wave frequency, period, and speed.

Exercise 3

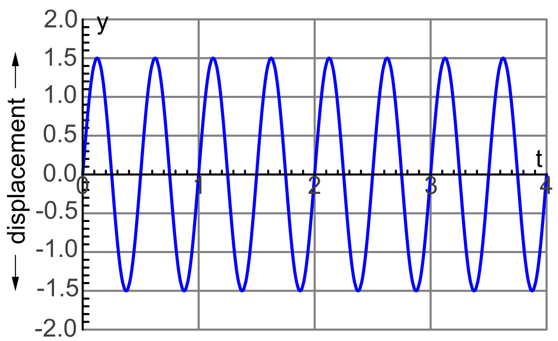
Match the sine functions with the correct graph.

a. $y = 1.5\sin(t)$

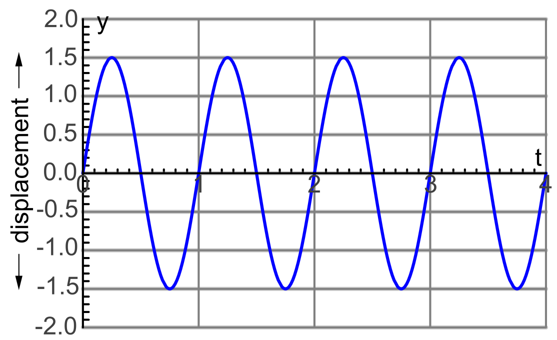
b. $y = \sin(2t)$

c. $y = 1.5\sin\left(\frac{t}{2}\right)$

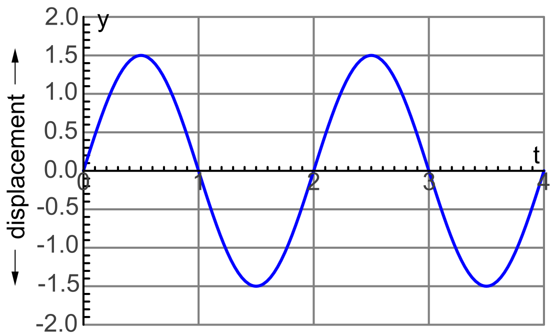
d. $= 1.5\sin(2t)$



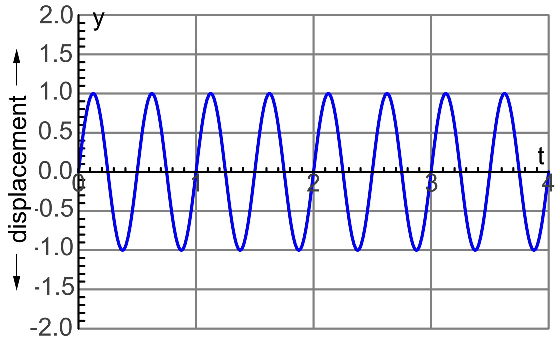
Graph 1



Graph 2



Graph 3



Graph 4

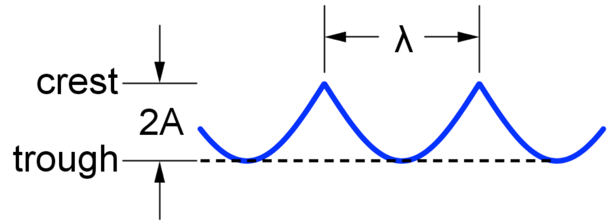
Exercise 4

When you ride a rotating Ferris wheel, your height above the ground rises to a maximum and then falls to a minimum. The Tenpozan Ferris Wheel in Osaka, Japan has a diameter of 100 meters. Write an equation and draw a graph of a rider's height (above the lowest point) as a function of the wheel's angle of rotation (in degrees).

Solutions to Practice Exercises

Solutions to Practice Exercises

1. The wave amplitude is half the total distance from crest to trough. The fisherman estimated that distance as one-fourth the boat height:
   14 18 inches = 4.5 inches. Therefore, the wave amplitude is half of 4.5 inches, or 2.25 inches.



   The fisherman sees 13 crests between the front and back of the 16-foot boat. That means there are 12 crest-to-crest (peak-to-peak) wavelengths in a span of 16 feet. So, we can calculate the length of one wave, or the wavelength .

$$\lambda = \frac{16 \text{ ft}}{12 \text{ cycles}} = 1.33 \text{ ft per cycle}$$

2. If 6 crests pass by, then the fisherman actually times the passing of 5 wavelengths (or cycles) in 10 seconds. So, the wave frequency f is

$$f = \frac{\text{number of cycles}}{\text{time interval}} = \frac{5 \text{ cycles}}{10 \text{ sec}} = 0.5 \text{ cycles per sec}$$

   The period T can be calculated from the frequency.

$$T = \frac{1}{f} = \frac{1}{0.5 \text{ cycles per sec}} = 2.0 \text{ sec per cycle}$$

And the wave speed is simply the product of the frequency and wavelength (from Exercise 1).

$$c = \lambda f = \left(1.33\frac{\text{ft}}{\cancel{\text{cycle}}}\right)\left(0.5\frac{\cancel{\text{cycles}}}{\text{sec}}\right) = 0.665\frac{\text{ft}}{\text{sec}}$$

3. a. $y = 1.5\sin(t)$: Graph 2 has an amplitude of 1.5, and a frequency of 1 (or period of 1).

   b. $y = \sin(2t)$: Graph 4 has an amplitude of 1, and a frequency of 2 (or period of $\frac{1}{2}$).

   c. $y = 1.5\sin\left(\frac{t}{2}\right)$: Graph 3 has an amplitude of 1.5, and a frequency of $\frac{1}{2}$ (or period of 2).

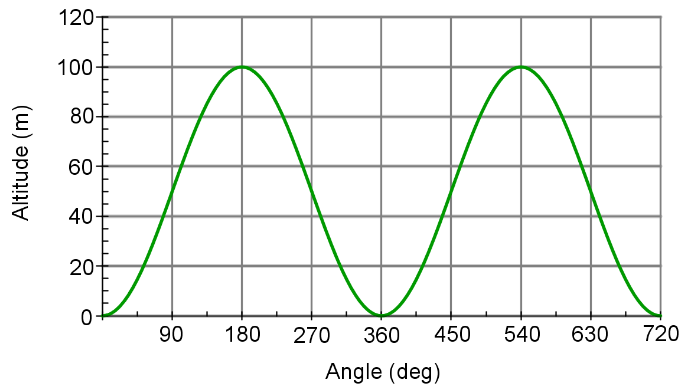
   d. $y = 1.5\sin(2t)$: Graph 1 has an amplitude of 1.5, and a frequency of 2 (or period of $\frac{1}{2}$).

4. A sketch of a rider's position on the circular wheel shows that height above ground (altitude) as a function of angle involves the cosine of an angle, where we will define the angle to be measured from the starting point at the rider's lowest point on the wheel. Notice the right-triangle that has the wheel radius R as its hypotenuse. The adjacent leg of the triangle is R cos θ, so that the rider's altitude h is

$$h = R - R\cos\theta$$

where h = altitude of rider
R = radius of wheel
$\theta = \frac{1}{2}$ angle

A graph of the altitude h versus angle shows the rider's changing altitude as the Ferris wheel turns. Our graph here shows the angle rotating through two complete turns. Notice that at 0 degrees, the rider is at an altitude of zero, and at 180 degrees the rider is at the high point of 100 meters.



If we wished to include the fact that the rider is actually 12.5 meters above ground at the lowest point, we would rewrite our function as follows:

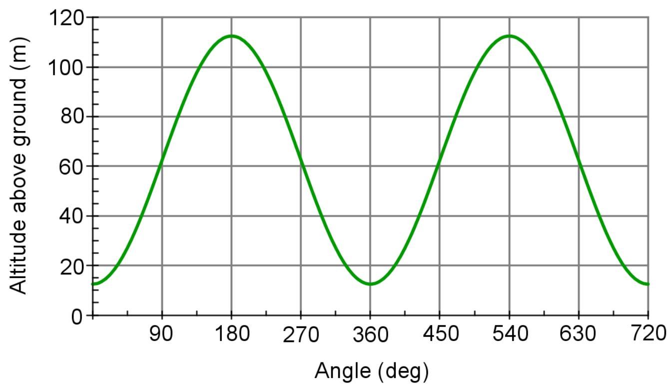
$$h = R - R\cos\theta + (12.5\text{ m})$$

where h = altitude of rider above ground

R = radius of wheel

θ = angle

and the graph changes only by shifting up 12.5 meters, as shown.



Career Video

Welding: A Skill Required In Every Industry

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Complex Numbers

13

Objectives

When you have completed this activity, you should be able to do the following:

1. Represent complex numbers as vectors on a complex number plane.
2. Represent complex numbers as phasors on a polar coordinate system.
3. Convert complex numbers between rectangular (a + bi) and polar forms ($Me^{i\theta}$).
4. Perform addition, subtraction, multiplication, and division with complex numbers.

Scenario

The impedance Z of an electric circuit is often represented in complex form:

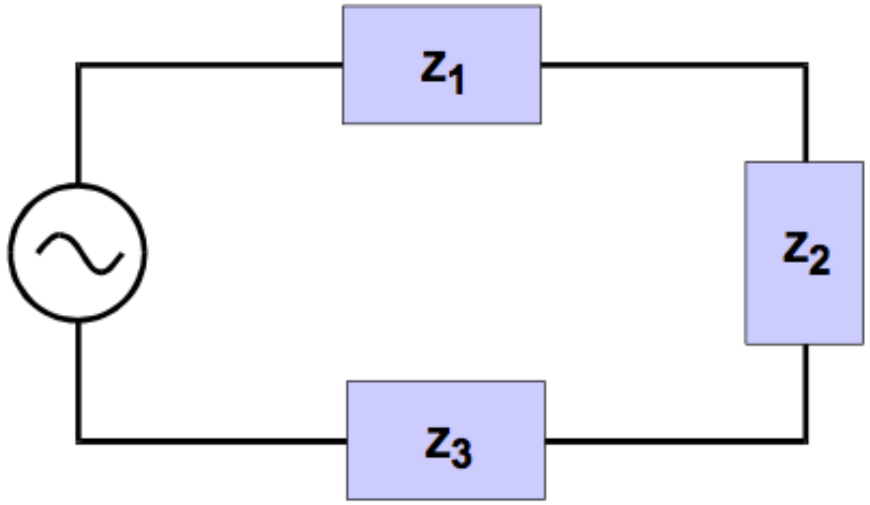
$$Z = R + iX$$

where R is the resistive (real) component and X is the reactive (imaginary) component. Three components are connected, with their impedances, as shown.

$Z1 = 40\ \Omega + i0\ \Omega$

$Z2 = 10\ \Omega - i0\ \Omega$

$Z3 = 18\ \Omega + i0\ \Omega$



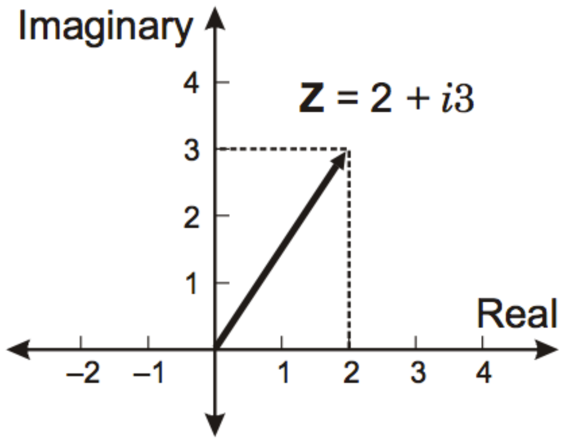
Find the total impedance of this series circuit.

Before looking at the solution, work through the lesson to further develop your skills in this area.

Complex Numbers

Complex Numbers Are Like Ordered Pairs

A complex number **Z** is similar to an ordered pair like (2, 3), but we show it as a sum of a "real" part and an "imaginary" part. The imaginary part is denoted by the "i" term.



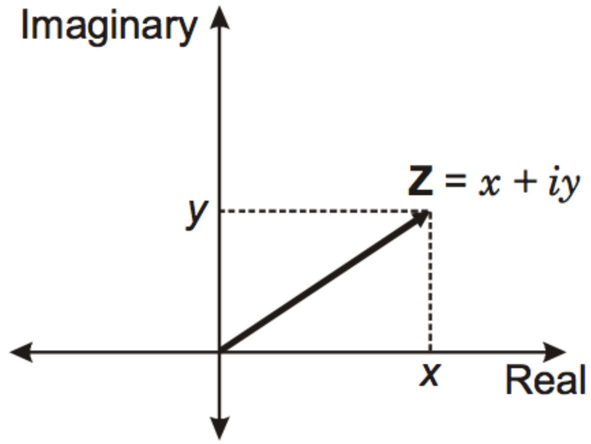
Z = 2 + i3

For the complex number **Z** above, 2 is the "real" part; 3 is the "imaginary" part.

Just as we plot ordered pairs (x, y) on a Cartesian coordinate plane, we plot complex numbers on a **complex number plane**. The horizontal axis is the "real" axis. The vertical axis is the "imaginary" axis. So, a plot of the complex

number **Z** = 2 + i3 would look like the graph above. We plot an "arrow" to indicate a complex vector quantity.

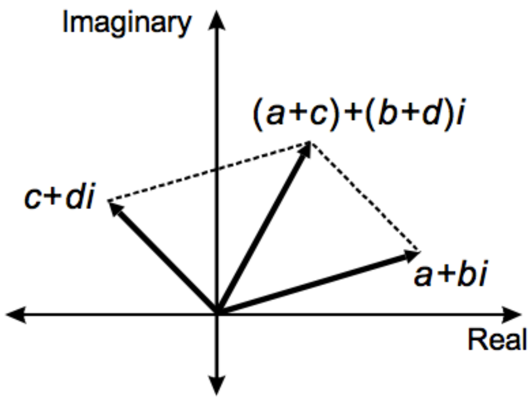
So, in general, any complex vector **Z** can be plotted on a complex number plane, as shown here.



Z = x + iy

Sums of Complex Numbers

The addition (or subtraction) of complex numbers is defined in a simple way: we sum the real parts and imaginary parts separately. So, for some value a, b, c, and d:



$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

The graphical representation of this sum corresponds to the adding of two vectors, as shown in the graph here.

Example 1

Find the sum of 2 + i3 and 4 – i:

$$(2+i3)+(4-i)=(2+4)+i(3-1)=6+i2$$

Products of Complex Numbers

The product of two complex numbers is a bit more complicated. This is because of the way that the "imaginary" number i is defined.

$i^2 = -1$

To find the product of two complex numbers, use the *distributive* property to multiply the terms in the two binomials.

$$(a+ib)\times(c+id)=a(c)+a(id)+ib(c)+ib(id)$$

Use distributive property.

$$(a+ib)\times(c+id)=(ac)+i(ad)+i(bc)+i^2(bd)$$

Regroup the terms.

$$(a+ib)\times(c+id)=(ac)+i(ad+bc)-(bd)$$

Combine imaginary terms and apply the definition of i^2 .

$$(a+ib)\times(c+id)=(ac-bd)+i(ad+bc)$$

Combine real terms.

Example 2

Find the product of 2 + i3 and 4 + i:

$$(2+i3)\times(4+i) = 2(4) + 2(i) + i3(4) + i3(i)$$

Use distributive property.

$$(2+i3)\times(4+i) = (2\cdot 4) + i(2\cdot 1 + 3\cdot 4) - (3\cdot 1)$$

Combine imaginary terms, and use definition of i^2 .

$$(2+i3)\times(4+i) = (8-3) + i(2+12)$$

$$= 5 + i14$$

Simplify.

Using Complex Conjugates when Dividing Complex Numbers

The **complex conjugate** of a complex number a + ib is *defined* to be a – ib. A typical use of conjugates is to simplify fractions that contain complex numbers in the denominator. We multiply by a unit fraction made with the **complex conjugate** of the denominator. The result is that the product's denominator is no longer complex, but real.

Example 3

Find the real and imaginary components of the complex fraction given by $Z = \frac{5-i}{1+i3}$.

To do this, multiply the numerator and the denominator by the complex conjugate of the denominator.

$$Z = \frac{5-i}{1+i3}$$ The given complex fraction.

$$Z = \frac{5-i}{1+i3} \times \frac{1-i3}{1-i3}$$ ← A unit fraction

Multiply by a unit fraction formed with the complex conjugate of the denominator of the given fraction.

$$Z = \frac{5(1) + 5(-i3) - i(1) - i(-i3)}{1(1) + 1(-i3) + i3(1) + i3(-i3)}$$

Use the distributive property to multiply terms in the numerator. Similarly for terms in the denominator.

$$Z = \frac{\left(5 + i^2 3\right) + i(-15 - 1)}{\left(1 - i^2 9\right) + i(-3 + 3)}$$

Apply the definition of i^2 . Collect the real terms and the imaginary terms and simplify.

$$Z = \frac{2 - i16}{10 + i0}$$

$$Z = \frac{2 - i16}{10}$$

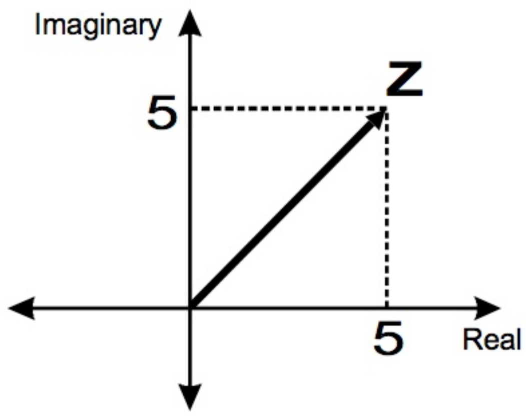
Denominator is no longer complex. Use distributive property to simplify.

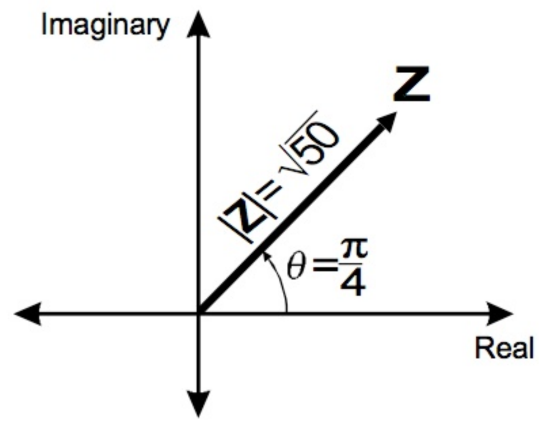
$$Z = 0.2 - i1.6$$

So, the real part of the given complex fraction is 0.2 and the imaginary part is –1.6.

Polar Representation of Complex Numbers

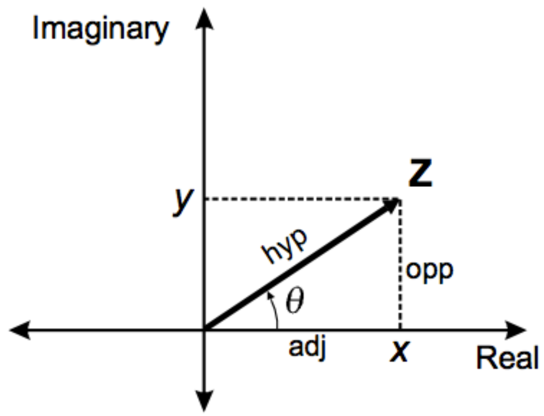
The previous examples used a *rectangular form* of complex numbers: having horizontal and vertical components. Complex numbers can also be represented in a *polar form*: as “phasors” having a *magnitude* and a *phase angle*.





The magnitude (Z) of a complex vector Z = x + iy is easily found with the Pythagorean formula:

$$|Z| = \sqrt{x^2 + y^2}$$

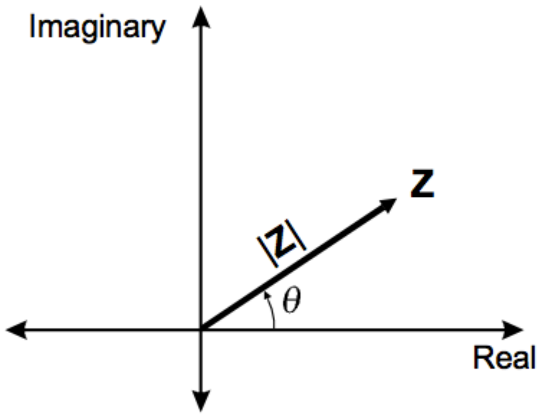


and the angle can be determined from any of the right triangle relationships.

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{|Z|} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{|Z|}$$

Using the above, we can then write Z in a different way:

$$Z = |Z|\cos\theta + i|Z|\sin\theta$$
$$= |Z|(\cos\theta + i\sin\theta)$$



In advanced mathematics courses, it is shown that $\cos\theta + i\sin\theta = e^{i\theta}$ (for θ in radians). Thus, we can also write in exponential form:

$$Z = |Z|e^{i\theta}$$

Multiplying and Dividing Complex Numbers

Multiplying and dividing complex vectors is relatively easy when we use the polar exponential form. For example, given two complex vectors $Y = |Y|e^{i\theta_Y}$ and $Z = |Z|e^{i\theta_Z}$, we can find the product $Y \times Z$.

$$Y \times Z = |Y|e^{i\theta_Y} \times |Z|e^{i\theta_Z}$$

The two given complex numbers, Y and Z, in polar exponential form.

$$Y \times Z = |Y| \times |Z| \left(e^{i\theta_Y}e^{i\theta_Z}\right)$$

Group the exponentials together.

$$Y \times Z = |Y| \times |Z| \; e^{i(\theta_Y + \theta_Z)}$$

Use rule for multiplying powers with same base:

$$x^a \bullet x^b = x^{a+b}$$

When given in polar form, to find the product of complex numbers we can multiply the magnitudes and sum the phase angles in the exponents, as shown above. In a similar way we can find the quotient $\frac{Y}{Z}$.

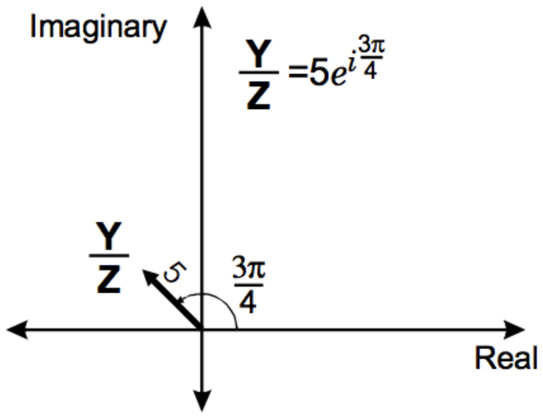
Example 4

Given $Y = 15e^{i\left(\frac{\pi}{2}\right)}$ and $Z = 3e^{-i\left(\frac{\pi}{4}\right)}$, find $Y \times Z$ and $\frac{Y}{Z}$:

$$Y \times Z = 15e^{i\left(\frac{\pi}{2}\right)} \times 3e^{-i\left(\frac{\pi}{4}\right)}$$

$$Y \times Z = (15 \times 3)e^{i\left(\frac{\pi}{2} - \frac{\pi}{4}\right)}$$

$$Y \times Z = 45e^{i\left(\frac{\pi}{4}\right)}$$

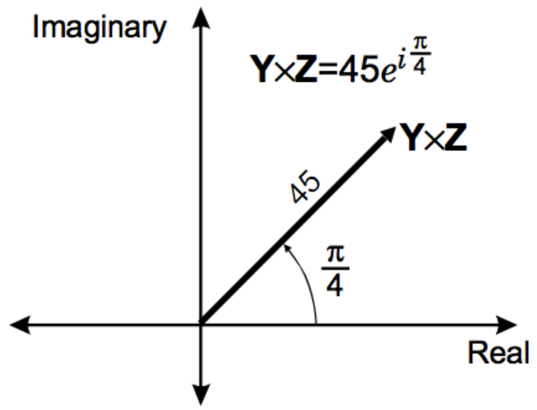


and

$$\frac{Y}{Z} = \frac{15e^{i\left(\frac{\pi}{2}\right)}}{3e^{-i\left(\frac{\pi}{4}\right)}}$$

$$\frac{Y}{Z} = \left(\frac{15}{3}\right)e^{i\left(\frac{\pi}{2} - \left(-\frac{\pi}{4}\right)\right)}$$

$$\frac{Y}{Z} = 5e^{i\left(\frac{3\pi}{4}\right)}$$



Example 5

The impedance of a circuit containing resistive, capacitive, and inductive components can be represented with complex number notation using the formula:

$$Z = R + i(X_L - X_C)$$

where Z = impedance, in ohms (or Ω)
R = resistance, in ohms (or Ω)
X_L = inductive reactance, in ohms (or Ω)
X_C = capacitive reactance, in ohms (or Ω)

Reactance is dependent not only on the component values, but also on the AC frequency in the circuit:

$$X_L = 2\pi fL \quad \text{and} \quad X_C = \frac{1}{2\pi fC}$$

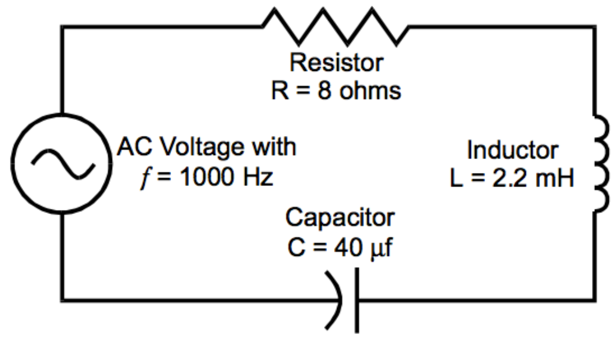
where f = the frequency, in Hertz (or cycles per second)

L = the inductance, in Henrys

C = the capacitance, in farads

To find the impedance of this RLC circuit, for an audio frequency of 1000 Hz, first calculate the reactances. Note that we must use units of Henrys and Farads, and so express the prefixes in scientific notation. (By definition, $1\,\text{ohm} = 1\,\text{H/sec} = 1\,\text{sec}/f$.)

$$X_L = 2\pi fL$$



$$X_L = 2\pi fL$$
$$X_L = 2\pi(1000\,\text{Hz})\left(2.2\times10^{-3}\,\text{H}\right)$$
$X_L = 13.8$ ohms, or $13.8\ \Omega$

$$X_C = \frac{1}{2\pi fC}$$
$$X_C = \frac{1}{2\pi(1000\ \text{Hz})\left(40\times10^{-6}\ \text{f}\right)}$$

$X_C = 4.0$ ohms, or $4.0\ \Omega$

Then the impedance of the circuit at 1000 Hz will be:

$$Z = R + i(X_L - X_C)$$

$$Z = (8\ \text{ohms}) + i(13.8\ \text{ohms} - 4.0\ \text{ohms})$$

$$Z = (8 + i9.8)\ \text{ohms}$$

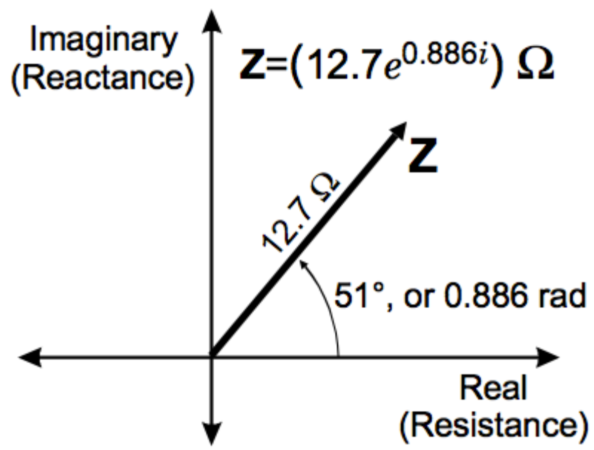
In polar form, we would have:

$$|Z| = \sqrt{(8\ \Omega)^2 + (9.8\ \Omega)^2} = 12.7\ \Omega$$

$$\theta = \tan^{-1}\left(\frac{9.8\ \Omega}{8\ \Omega}\right) = 0.886\ \text{rad, or } 51°$$

$$|Z| = (12.7e^{0.886i})\Omega$$

Notice that we can see that the real axis represents a pure resistance and the imaginary axis represents a pure reactance (a combination of inductance and capacitance).



Solution to Scenario Question

Solution to Scenario Question

The total impedance of a series of components is the sum of the individual impedances. The impedances Z_1 , Z_2 , and Z_3 are given in complex form—making it straightforward to find their sum mathematically.

$$\begin{aligned} Z_1 + Z_2 + Z_3 &= (40\ \Omega + i0\ \Omega) + (10\ \Omega - i95\ \Omega) + (18\ \Omega + i77\ \Omega) \\ &= (40\ \Omega + 10\ \Omega + 18\ \Omega) + i(0\ \Omega - 95\ \Omega + 77\ \Omega) \\ &= 68\ \Omega - i18\ \Omega \end{aligned}$$

Practice Exercises

Practice Exercises

Exercise 1

Find the sum of the complex numbers **A** = –7 + i3 and **B** = 8 + i2 both algebraically and graphically.

Exercise 2

Find the product of **A** and **B** (given in Exercise 1).

Exercise 3

Use the complex conjugate of **B** (use **A** and **B** found in Exercise 1) to evaluate the quotient $\frac{\mathbf{A}}{\mathbf{B}}$.

Exercise 4

Represent the complex numbers **A** and **B** (given in Exercise 1) in polar form. (**Hint:** Sketch the complex vector to verify the resulting polar angle.)Evaluate the product of these two vectors, convert the result back to rectangular form, and compare to the result of Exercise 2.

Exercise 5

Use the polar forms of **A** and **B** (found in Exercise 4) to evaluate the quotient $\frac{\mathbf{A}}{\mathbf{B}}$. Convert the result back to rectangular form and compare to the result of Exercise 3.

Exercise 6

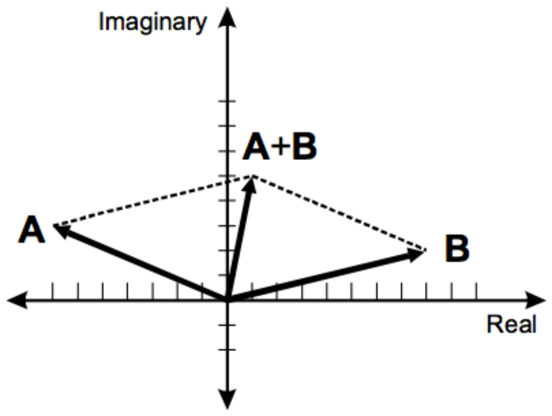
Find the complex impedance of the circuit in Example 4 for frequencies of 150 Hz, 2000 Hz, and 5000 Hz. Calculate the magnitude of the impedance vector at each frequency.

Solutions to Practice Exercises

Solutions to Practice Exercise

1. $A + B = (-7 + i3) + (8 + i2)$

$A + B = (-7 + 8) + i(3 + 2)$



2. $A \times B = (-7 + i3) \times (8 + i2)$

$A \times B = (-7 \cdot 8) + i(-7 \cdot 2 + 3 \cdot 8) - (3 \cdot 2)$

$A \times B = (-56 - 6) + i(-14 + 24)$

$A \times B = -62 + i10$

3. $\frac{A}{B} = \frac{-7 + i3}{8 + i2}$

$\frac{A}{B} = \frac{-7 + i3}{8 + i2} \times \frac{8 - i2}{8 - i2}$

$$\frac{A}{B} = \frac{(-7) \bullet 8 + i\left((-7) \cdot (-2) + 3 \cdot 8\right) - 3 \cdot (-2)}{8 \cdot 8 + i\left(8 \cdot (-2) + 2 \cdot 8\right) - 2 \cdot (-2)}$$

$$\frac{A}{B} = \frac{(-56 + 6) + i(14 + 24)}{(64 + 4) + i(-16 + 16)}$$

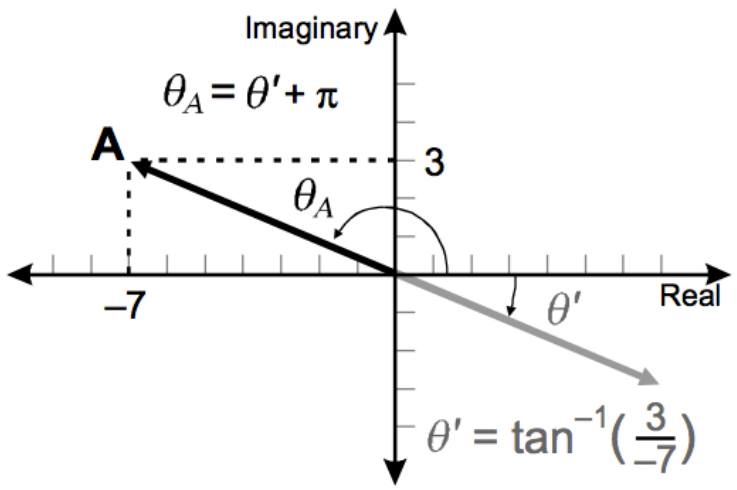
$$\frac{A}{B} = \frac{-50 + i38}{68}$$

$$\frac{A}{B} = -0.74 + i0.56 \text{ (rounded)}$$

4. For, $A = -7 + i3$

$$|A| = \sqrt{(-7)^2 + 3^2} = 7.62$$

$$\theta_A = \tan^{-1}\left(\frac{3}{-7}\right) = -0.405 \text{ radians}$$



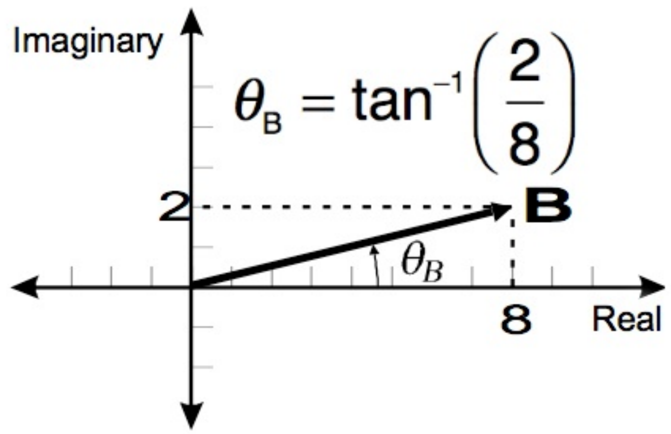
The function $\tan^{-1}(\frac{y}{x})$ needs our assistance for angles that we know lie in "negative x territory," as is the case here. Since we know from the sketch that θ_A must be in Quadrant II, we adjust the $\tan^{-1}$ result by **adding** π.

$$\theta_A = -0.405 + \pi = 2.737 \text{ radians}$$

Similarly, for $B = 8 + i2$

$$|B| = \sqrt{8^2 + 2^2} = 8.25$$

$$\theta_B = \tan^{-1}\left(\frac{2}{8}\right) = 0.2450 \text{ radians}$$



Here, the result correctly points to Quadrant I.

So, then $A = -7 + i3 = 7.62 \bullet e^{i2.737}$ and $B = 8 + i2 = 8.25 \bullet e^{i0.2450}$. And the product is found as follows.

$$A \times B = \left(7.62 \cdot e^{i2.737}\right) \times \left(8.25 \cdot e^{i0.2450}\right)$$

$$A \times B = \left(7.62 \cdot 8.25\right) \times \left(e^{i2.737+i0.2450}\right)$$

$$A \times B = 62.9 \cdot e^{i2.982}$$

To convert back to rectangular form…

$$A \times B = 62.9 \cdot e^{i2.982}$$

$$A \times B = 62.9 \cdot \left[\cos(2.982) + i\sin(2.982)\right]$$

$$A \times B = 62.9 \cdot \left[-0.9873 + i0.1589\right]$$

$A \times B = -62 + i10$ (rounded)

Which agrees with Exercise 2.

5. Using the results from Exercise 4, we have $A = -7 + i3 = 7.62 \cdot e^{i2.737}$ and $B = 8 + i2 = 8.25 \cdot e^{i0.2450}$. The quotient can then be evaluated:

$$\frac{A}{B} = \frac{7.62 \cdot e^{i2.737}}{8.25 \cdot e^{i0.2450}}$$

$$\frac{A}{B} = \frac{7.62}{8.25} \cdot e^{i2.737-i0.2450}$$

$$\frac{A}{B} = 0.9236 \cdot e^{i2.492}$$

Career Video

Mapping The Past And The Present Through GIS

The "classroom" could be any place where maps are used. In this episode, we take a walk outside with an instructor and students who are plotting maps to help one community preserve history.

Appendix

Quick Reference of Greek Letters, Constants and Unit Descriptions

Greek Letters

α	Α	Alpha
β	Β	Beta
γ	Γ	Gamma
δ	Δ	Delta
ε	Ε	Epsilon
ζ	Ζ	Zeta
η	Η	Eta
θ	Θ	Theta

ι	Ι	Iota
κ	Κ	Kappa
λ	Λ	Lambda
μ	Μ	Mu
ν	Ν	Nu
ξ	Ξ	Xi
ο	Ο	Omicron
π	Π	Pi

ρ	Ρ	Rho
σ	Σ	Sigma
τ	Τ	Tau
υ	Υ	Upsilon
ϕ, φ	Φ	Phi
χ	Χ	Chi
ψ	Ψ	Psi
ω	Ω	Omega

Units

Base Units		
Quantity	**Name**	**Symbol**
Length	Meter	m
Time	Second	s
Mass	Kilogram	kg
Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol

Supplementary Units		
Quantity	**Name**	**Symbol**
Plane Angle	Radian	rad
Solid Angle	Steradian	sr

Physical Constants

Name	Symbol	Approximate Value
Speed of light in vacuum	c	2.997×10^{8} m/s
Permeability of vacuum	μ_o	$4\pi \times 10^{-7}$ N/A²
Permittivity of vacuum	ε_o	8.85×10^{-12} C²/N • m²
Gravitational constant	g	9.81 m/s²
Avogadro's constant	N_A	6.02×10^{23} mol⁻¹
Boltzmann's constant	k	1.38×10^{-23} J/K
Stefan-Boltzmann's constant	σ	5.67×10^{-8} W/m² • K⁴
Planck's constant	h	6.63×10^{-34} J • s
Elementary charge	e	1.60×10^{-19} C
Electron rest mass	m_e	9.11×10^{-31} kg
Proton rest mass	m_p	1.67×10^{-27} kg
Neutron rest mass	m_n	1.67×10^{-27} kg
Atomic mass unit	u	1.66×10^{-27} kg

Derived Units

Unit	Symbol	Unit Details
Newton	N	$kg \cdot m/s^2$
Joule	J	$N \cdot m = kg \cdot m^2/s^2$
Watt	W	$J/s = kg \cdot m^2/s^3$
Hertz	Hz	1/s
Coulomb	C	$A \cdot s$
Volt	V	$J/C = W/A = kg \cdot m^2/(A \cdot s^3)$
Ohm	Ω	$V/A = kg \cdot m^2/(A^2 \cdot s^3)$
Tesla	T	$N/(A \cdot m) = kg/(A \cdot s^2)$
Weber	Wb	$T \cdot m^2 = V \cdot s = J/A = kg \cdot m^2/(A \cdot s^2)$
Henry	H	$V \cdot s/A = \Omega \cdot s = kg \cdot m^2/(A^2 \cdot s^2)$
Farad	F	$C/V = s/\Omega = A^2 \cdot s^4/(kg \cdot m^2)$

SI Prefixes

Multiplication Factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h*
10^{1}	deka	da*
10^{-1}	deci	d*
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

*Avoid these prefixes when possible.

Common Identities

Powers and Roots		Exponents and Logarithms
• $b^{\frac{x}{y}} = \sqrt[y]{b^x}$		• $\log_b x = y \Leftrightarrow b^y = x$
• $b^x \cdot b^y = b^{x+y}$	↔	• $\log_b(xy) = \log_b x + \log_b y$
• $\dfrac{b^x}{b^y} = b^{x-y}$	↔	• $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
• $(b^x)^y = b^{xy}$		• $\log_b x^y = y\log_b x$
• $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$		• $\log_{10} x = \log x$
• $b^{-x} = \dfrac{1}{b^x}$		• $\log_e x = \ln x$
• $\dfrac{1}{b^{-x}} = b^x$		• $\log_b x = \dfrac{\ln x}{\ln b} = \dfrac{\log x}{\log b}$
• $b^x = b^y \Leftrightarrow x = y$	↔	• $\log_b x = \log_b y \Leftrightarrow x = y$
• $b^0 = 1$	↔	• $\log_b 1 = 0$
• $b^1 = b$	↔	• $\log_b b = 1$
		• $\log_b b^x = x = b^{\log_b x}$

Trigonometry

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- $c^2 = a^2 + b^2 - 2ab \cdot \cos C$
- $b^2 = a^2 + c^2 - 2ac \cdot \cos B$
- $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Tips for Scientific Calculators

Button	Example	Button Sequence	Answer
sin, cos, tan	cos 60°	[6] [0] [cos]	0.5
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$	$\theta = \sin^{-1}(1)$	[1] [2nd or INV] [sin]	90°
x^2	27^2	[2] [7] [x^2]	729
$\sqrt{x}$ or $\sqrt{\ }$	$\sqrt{3}$	[3] [2nd or INV] [$\sqrt{x}$]	1.73
y^x	$53^{1.7}$	[5] [3] [y^x] [1] [.] [7]	853.6
$\sqrt[x]{y}$	$\sqrt[3]{5}$	[5] [2nd or INV] [$\sqrt[x]{y}$] [3] [=]	1.71

Button	Example	Button Sequence	Answer
EE or Exp or E	8.6×10^4	[8] [.] [6] [EE] [4]	—
log	log 17	[1] [7] [log]	1.23
ln *x* or ln	ln 2.4	[2] [.] [4] [ln x]	0.88
(,)	$5 \times (3 + 2)$	[5] [×] [(] [3] [+] [2] [)] [=]	25
π	2π	[2] [×] [π] [=]	6.28
+/– or (–)	$9 \times (-6)$	[9] [×] [6] [+/–] [=]	–54
	3.8×10^{-3}	[3] [.] [8] [EE] [3] [+/–]	—

Conversion Tables

The following tables list the common conversions among various units of measure.

LENGTH

Length	*m*	*km*	*inch*	*ft*	*mi*
1 meter	1	1.0×10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer	1000	1	3.937×10^{4}	3281	0.6214
1 inch	0.0254	2.54×10^{-5}	1	0.0833	1.578×10^{-5}
1 foot	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1609	1.609	6.336×10^{4}	5280	1

1 angstrom = 10^{-10} m
1 micron (μ) = 10^{-6} m
1 rod = 16.5 ft
1 league = 3 nautical miles
1 nautical mile = 1852 m = 1.1508 mi = 6076.10 ft
1 fathom = 6 ft
1 yard (yd) = 3 ft
1 mil = 10^{-3} inch

TIME

Time	*yr*	*day*	*h*	*min*	*s*
1 year	1	3.652×10^{2}	8.766×10^{3}	5.259×10^{5}	3.156×10^{7}
1 day	2.738×10^{-3}	1	24	1.44×10^{3}	8.64×10^{4}
1 hour	1.141×10^{-4}	4.167×10^{-2}	1	60	3.6×10^{3}
1 minute	1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 second	3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

1 day = period of rotation of earth = 86,164 s
1 year = period of revolution of earth = 365.242 days

MASS

Mass	*g*	*kg*	*lbm*	*slug*	*ton-mass*
1 gram	1	1.0×10^{-3}	2.205×10^{-3}	6.852×10^{-5}	1.102×10^{-6}
1 kilogram	1.0×10^{3}	1	2.205	6.852×10^{-2}	1.102×10^{-3}
1 pound mass	4.536×10^{2}	0.4536	1	3.108×10^{-2}	5.0×10^{-4}
1 slug	1.459×10^{4}	1.459×10^{1}	3.217×10^{1}	1	1.609×10^{-2}
1 ton-mass	9.072×10^{5}	9.07×10^{2}	2.0×10^{3}	6.216×10^{1}	1

1 long ton = 2240 lb = 20 cwt
1 hundredweight (cwt) = 112 lb
1 metric ton = 1000 kg = 2205 lb
1 stone = 14 lb
1 carat = 0.2 g

FORCE

Force	*dyne*	*kgf*	*N*	*lb*	*pdl*
1 dyne	1	1.020×10^{-6}	1.0×10^{-5}	2.248×10^{-6}	7.233×10^{-5}
1 kilogram force	9.807×10^{5}	1	9.807	2.205	70.93
1 newton	1.0×10^{5}	0.1020	1	0.2248	7.233
1 pound	4.448×10^{5}	0.4536	4.448	1	32.17
1 poundal	1.383×10^{4}	1.410×10^{-2}	0.1383	3.108×10^{-2}	1

PRESSURE

Pressure	*atm*	*inch of water*	*cm Hg*	*N/m²*	*lb/inch² (psi)*
1 atmosphere	1	4.068×10^{2}	7.6×10^{1}	1.013×10^{5}	1.470×10^{1}
1 inch of water[a]	2.458×10^{-3}	1	0.1868	2.491×10^{2}	3.613×10^{-2}
1 cm of mercury[a]	1.316×10^{-2}	5.353	1	1.333×10^{3}	0.1934
1 newton per square meter	9.869×10^{-6}	4.105×10^{-3}	7.501×10^{-4}	1	1.450×10^{-4}
1 pound per square inch	6.805×10^{-2}	2.768×10^{1}	5.171	6.895×10^{3}	1

[a]Under standard gravitational acceleration and temperature of 4°C for water, 0°C for mercury

1 bar = 1.013×10^{5} N/m²
1 cm of water = 98.07 N/m²
1 torr = 1 mm of Hg
1 ft of water = 62.43 lb/ft²
1 atm = 29.92 in. of Hg
1 atm = 33.92 ft of water
1 atm = 2117 lb/ft²

ENERGY

Energy	*Btu*	*ft·lb*	*J*	*kcal*	*kWh*
1 British thermal unit	1	7.779×10^{2}	1.055×10^{3}	0.2520	2.930×10^{-4}
1 foot-pound	1.285×10^{-3}	1	1.356	3.240×10^{-4}	3.766×10^{-7}
1 joule	9.481×10^{-4}	0.7376	1	2.390×10^{-4}	2.778×10^{-7}
1 kilocalorie	3.968	3.086×10^{3}	4.184×10^{3}	1	1.163×10^{-3}
1 kilowatt-hour	3.413×10^{3}	2.655×10^{6}	3.6×10^{6}	8.602×10^{2}	1

1 Btu = 252 cal
1 Btu = 778 ft • lb
1 Btu = 1055 J
1 J = 0.239 cal
1 cal = 3.09 ft • lb
1 cal = 4.18 J
1 kcal = 1000 cal
1 ft • lb = 0.324 cal

MASS-ENERGY EQUIVALENTS

Mass-Energy Equivalents	*kg*	*amu*	*J*	*MeV*
1 kilogram	1	6.02×10^{26}	8.987×10^{16}	5.610×10^{29}
1 atomic mass unit	1.666×10^{-27}	1	1.492×10^{-10}	9.315×10^{2}
1 joule	1.113×10^{-17}	6.68×10^{9}	1	6.242×10^{12}
1 million electron volts	1.783×10^{-30}	4.17×10^{22}	1.602×10^{-13}	1

POWER

Power	*Btu/h*	*ft·lb/s*	*hp*	*kcal/s*	*W*
1 Btu/h	1	0.2161	3.929×10^{-4}	7.0×10^{-5}	0.2930
1 ft·lb/s	4.628	1	1.818×10^{-3}	3.239×10^{-4}	1.356
1 horsepower	2.545×10^{3}	5.50×10^{2}	1	0.1782	7.457×10^{2}
1 kcal/s	1.429×10^{4}	3.087×10^{3}	5.613	1	4.184×10^{3}
1 watt	3.413	0.7376	1.341×10^{-3}	2.390×10^{-4}	1

1 ton refrigeration = 12,000 Btu/h

SPEED

Speed	*ft/s*	*km/h*	*m/s*	*mi/h*	*knot*
1 foot per s	1	1.097	0.348	0.6818	0.5925
1 kilometer per h	0.9113	1	0.2778	0.6214	0.5400
1 meter per s	3.281	3.6	1	2.237	1.944
1 mile per h	1.467	1.609	0.4470	1	0.8689
1 knot	1.688	1.852	0.5144	1.151	1

1 knot = 1 nautical mile/h

WAVELENGTH CONVERSIONS

λ	Å (angstrom)	nm (nanometer)	μm (micrometer)	cm (centimeter)	m (meter)
1 Å	1	10^{-1}	10^{-4}	10^{-8}	10^{-10}
1 nm	10	1	10^{-3}	10^{-7}	10^{-9}
1 μm	10^{4}	10^{3}	1	10^{-4}	10^{-6}
1 cm	10^{8}	10^{7}	10^{4}	1	10^{-2}
1 m	10^{10}	10^{9}	10^{6}	10^{2}	1